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MEÝDAN NAZARYÝETI

Ýokary okuw mekdepleri üçin okuw gollanmasy

Türkmenistanyň Bilim ministrligi
tarapyndan hödürlenildi

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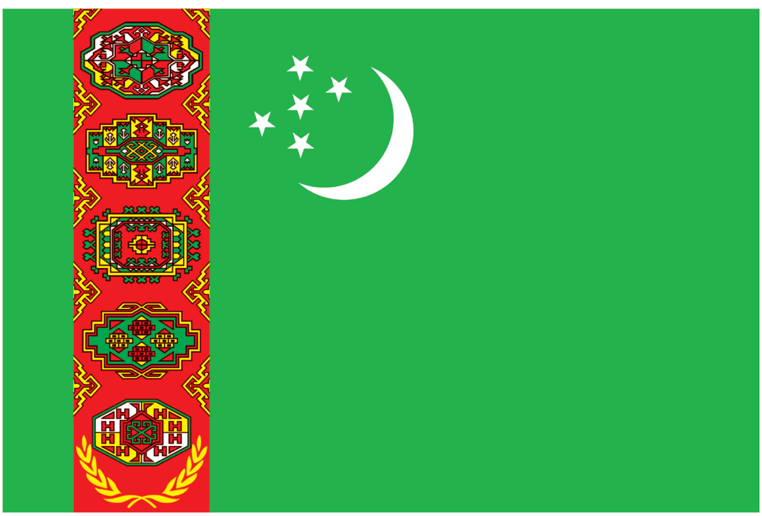
GURBANGULY BERDIMUHAMEDOW

TÜRKMENISTANYŇ DÖWLET TUGRASY



TÜRKMENISTANYŇ DÖWLET BAÝDAGY

TÜRKMENISTANYŇ DÖWLET SENASY

Janym gurban saňa, erkana ýurdum,
Mert pederleň ruhy bardyr köňülde.
Bitarap, garaşsyz topragyň nurdur,
Baýdagyň belentdir dünýäň öňünde.

Gaýtalama:
Halkyň guran Baky beýik binasy,
Berkarar döwletim, jigerim-janym.
Başlaryň täji sen, diller senasy,
Dünýä dursun, sen dur, Türkmenistanym!

Gardaşdyr tireler, amandyr iller,
Owal-ahyr birdir biziň ganymyz.
Harasatlar almaz, syndyrmaz siller,
Nesiller döş gerip gorar şanymyz.

Gaýtalama:
Halkyň guran Baky beýik binasy,
Berkarar döwletim, jigerim-janym.
Başlaryň täji sen, diller senasy,
Dünýä dursun, sen dur, Türkmenistanym!

SÖZBAŞY

Galkynyş zamanynyň inženeriniň ýokary matematikadan ýeterlik taýýarlygynyň bolmaklygy, ylmyň we tehnikanyň dürli ugurlarynda uly ösüşlere alyp barýar. Ol ösüşleriň kämilleşmegine ýeterlik ýardam edýän ýokary matematikanyň esasy bölümleriniň biri bolan meýdan nazaryýeti bölümidir.

Ýokary okuw mekdepleriniň okuw meýilnamalaryna girizilen matematikanyň şol bölümini düýpli özleşdirmeklik üçin niýetlenen bu okuw gollanmasy ýeterlik derejede nazary materiallary, köp sanly amaly mysallary we meseleleri öz içine alýar. Gollanmada ulanylýan mysallaryň we meseleleriň 54-si işlenip görkezildi we 144-si bolsa, özbaşdak işlemeklige hödürlenildi.

Gollanma belliklerini we maslahatlaryny ýollan okyjylara awtorlar öz minnetdarlygyny bildirýär.

I BÖLÜM

§ 1. Skalýar meýdan

Goý, D giňişligiň erkin ýaýlasy bolsun.

Kesgitleme. Eger şu ýaýlanyň her bir M nokadyna haýsy hem bolsa $u(M)$ san degişli bolsa, onda şu ýaýlada skalýar meýdan berlen diýilýär. Görnüşi ýaly skalýar meýdan tutuş giňişlik ýa-da giňişligiň belli bir bölegi bolup biler.

Skalýar meýdanlaryň mysallary bolup birmeňzeş gyzdyrylmadyk jisimiň nokatlarynyň temperaturasy (bu jisimiň her bir M nokadyna $u(M)$ temperatura degişlidir), elektrik meydanyň potensialy we ş.m. hyzmat edýär.

Eger $u(M)$ diňe M nokada bagly bolsa, onda skalýar meýdana stasionar meýdan, eger M nokada we t wagta bagly bolsa, meýdan stasionar bolmadyk meýdan diýip kabul edilýär.

Skalýar meýdan $Oxyz$ koordinatalar ulgamyna degişli bolsa, M nokadyň berilmegi, onuň x, y, z koordinatalarynyň berilmegi bilen deňgüýçlüdir. Şonuň üçin $u(M)$ skalýar funksiýanyň berilmegi, $u(x,y,z)$ funksiýanyň berilmegini aňladýar. Şunlukda, biz üç üýtgeýän ululykly funksiýanyň fiziki düşündirilişini aldyk.

Kesgitleme. Skalýar meýdanyň dereje üsti $u(M)$ funksiýanyň hemişelik C sana deň bolan nokatlarynyň köplügidir. Şonuň üçin dereje üstüň deňlemesini aşakdaky görnüşde ýazyp bileris:

$$u(x,y,z)= C. \qquad (1)$$

Fizikada potensial meydanyň dereje üstüne ekwipotensial (deňpotensially) üst diýilýär. $M_0(x_0,y_0,z_0)$ nokat arkaly geçýän dereje üstüň deňlemesi aşakdaky görnüşde ýazylýar:

$$u(x,y,z)= u_0(x_0,y_0,z_0).$$

Eger skalýar meýdan tekizlikde berlen bolsa, onda u funksiýa iki x we y ululyklara baglydyr. Onuň dereje çyzygynyň deňlemesi

$$\boldsymbol{u(x,y)=C}$$

görnüşde ýazylýar.

1-nji mysal. $y=\arcsin\frac{z}{\sqrt{x^2+y^2}}$ skalýar meýdanyň dereje üstüni tapmaly.

Çözülişi. Berlen skalýar meýdanyň barlyk ýaýlasy

$$\left|\frac{z}{\sqrt{x^2+y^2}}\right| \le 1 \text{ ýa-da } 0 \le \frac{z^2}{x^2+y^2} \le 1$$

deňsizliklerden tapylýar. Bu ýerden $0\le z^2\le x^2+y^2$. Görnüşi ýaly meýdan $z^2=x^2+y^2$ tegelek konusyň daşynda we onuň özünde kesgitlenendir. Onuň $O(0,0,0)$ depesi bu ýaýla degişli däldir.

Dereje üst

$$\arcsin\frac{z}{\sqrt{x^2+y^2}} = C, -\frac{\pi}{2} \le C \le \frac{\pi}{2}$$

deňlik bilen tapylýar.

$\frac{z}{\sqrt{x^2+y^2}} = \sin C, z^2 = (x^2+y^2)\sin^2 C$. Bu bolsa $z^2=x^2+y^2$ konusyň daşyndaky tegelek konuslaryň maşgalasydyr.

2-nji mysal. $u=x^2-y^2$ skalýar meýdanyň dereje çyzygyny tapmaly.

Çözülişi. Dereje çyzygy $x^2-y^2=C$, $C=const$ deňleme bilen kesgitlenýär. Eger $C=0$ bolsa $y=x$, $y=-x$ iki göni çyzygy alýarys. $C\neq 0$ bolsa meýdanyň dereje çyzygy giperbolalaryň maşgalasydyr.

Aşakdaky skalýar meýdanlaryň dereje üstlerini tapmaly.

1. $u - \frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} + \frac{(z-z_0)^2}{c^2}$;

2. $u = x^2+y^2+z^2$;

3. $u=x+2y+3z$;

4. $u=x^2+y$;

5. $u = \frac{x^2+y^2}{z}$;

6. $u=3^{x+2y-z}$.

Aşakdaky skalýar meýdanlaryň dereje çyzyklaryny tapmaly.

7. $u=2x-y$; 8. $u = \ln\sqrt{\dfrac{y}{2x}}$;

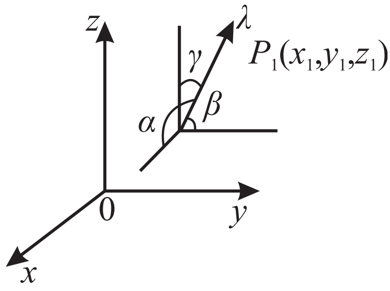
9. $u = \dfrac{y^2}{x}$; 10. $u = e^{x^2-y^2}$.

§ 2. Ugur boýunça önüm

Kesgitleme. Eger $u=u(x,y,z)$, $A(P)=L(x,y,z)i+Q(x,y,z)j+R(x,y,z)k$ skalýar we wektor meýdanlar bolsa we $u(x,y,z)$, $L(x,y,z)$, $Q(x,y,z)$, $R(x,y,z)$ funksiýalaryň n-nji tertipli hususy önümleri bar bolsa, onda ol meýdanlara n gezek differensirlenýän meýdanlar diýilýär.

Skalýar meýdanlar öwrenilen mahalynda esasy düşünjeleriň biri bolan funksiýanyň berlen ugur boýunça önümi düşünjesidir. Ol şol funksiýanyň berlen ugur boýunça üýtgeýiş tizligini görkezýär.

Goý, skalýar meýdan, ýagny $u(x,y,z)$ funksiýa berlen bolsun. $P(x,y,z)$ nokady alyp, ondan çykýan λ şöhlä garalyň. Ol şöhle koordinata oklary bilen α, β, γ burçlary emele getirýän bolsun. e_λ bolsa λ şöhle bilen ugurdaş birlik wektor bolup, onuň proýeksiýalary e_λ {cosα, cosβ, cosγ} bolar. Goý, $P(x_1, y_1, z_1)$ nokat λ şöhlede ýatýan bolsun.



1-nji surat

$|PP_1| = \rho \cdot PP_1$ wektoryň koordinata oklara bolan proýeksiýalary, birinjiden $p\cos\alpha$, $p\cos\beta$, $p\cos\gamma$, ikinjiden bolsa x_1-x, y_1-y, z_1-z tapawutlara deňdir. Diýmek,

$$x_1 = x + \rho\cos\alpha, y_1 = y + \rho\cos\beta, z_1 = \rho\cos\gamma.$$

Funksiýanyň P nokatdan P_1 nokada geçendäki artdyrmasyna garalyň:

$$u(P_1) - u(P) = u(x + \rho\cos\alpha, y + \rho\cos\beta, z + \rho\cos\gamma) - u(P).$$

$$\frac{u(P_1) - u(P)}{\rho}$$

gatnaşygy düzeliň we $\rho \to 0$ bolanda predele geçeliň:

$$\lim_{\rho\to 0}\frac{u(P_1) - u(P)}{\rho};$$

$$\lim_{\rho\to 0}\frac{u(x + \rho\cos\alpha, y + \cos\beta, z + \cos\gamma) - u(P)}{\rho}.$$

Eger-de şu predel bar bolsa we tükenikli bolsa, onda oňa $u(x,y,z)$ funksiýadan λ ugur boýunça alnan önüm diýilýär we $\frac{\partial u}{\partial \lambda}$ ýa-da $u'_\lambda(x,y,z)$ bilen belgilenýär.

Eger λ ugur Ox okuň položitel ugry bilen gabat gelse, onda $\alpha = 0, \beta = \gamma = \frac{\pi}{2}$ we $\lim\limits_{\rho\to 0}\frac{u(x+\rho, y, z) - u(x,y,z)}{\rho} = \frac{\partial u}{\partial x}$.

Şeýle hem $\beta = 0, \alpha = \gamma = \frac{\pi}{2}$ we $\gamma = 0, \beta = \alpha = \frac{\pi}{2}$ bolanda ýazyp bileris:

$$\lim_{\rho\to 0}\frac{u(x, y+\rho, z) - u(x,y,z)}{\rho} = \frac{\partial u}{\partial y};$$

$$\lim_{\rho\to 0}\frac{u(x, y, z+\rho) - u(x,y,z)}{\rho} = \frac{\partial u}{\partial z}.$$

$u'_x(x,y,z)$, $u'_y(x,y,z)$, $u'_z(x,y,z)$ önümler koordinata oklaryň ugruna görä $u(x,y,z)$ funksiýanyň üýtgeýiş tizligini aňladýar.

$u'_\lambda(x,y,z)$ bolsa funksiýanyň λ şöhläniň ugry boýunça P nokatdaky üýtgeýiş tizligini aňladýar.

Berlen ugur boýunça önümi hasaplamak aşakdaky teoremanyň kömegi bilen amala aşyrylýar.

Teorema. Eger $u(x,y,z)$ funksiýa differensirlenýän bolsa, onda erkin λ ugur boýunça önüm bardyr we ol aşakdaky formula bilen hasaplanylýar:

$$\frac{\partial u}{\partial \lambda} = \frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\cos\beta + \frac{\partial u}{\partial z}\cos\gamma. \qquad (2)$$

Bu ýerde $\cos\alpha$, $\cos\beta$, $\cos\gamma$, λ şöhläniň ugrukdyryjy kosinuslarydyr.

Subudy. Şerte görä $u(x,y,z)$ funksiýa differensirlenýär we onuň doly artdyrmasyny aşakdaky görnüşde ýazyp bileris:

$$\Delta u = u(x+\Delta x, y+\Delta y, z+\Delta z) - u(x,y,z) = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z + \varepsilon.$$

Bu ýerde $\varepsilon, \rho = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$ ululyga görä tükeniksiz kiçi ululykdyr.

$\Delta x = \rho\cos\alpha, \Delta y = \rho\cos\beta, \Delta z = \rho\cos\gamma$ deňligi ulanyp

$\Delta u = u(P_1) - u(P)$ tapawudy aşakdaky görnüşde ýazyp bileris:

$$u(P_1) - u(P) = \frac{\partial u}{\partial x}\rho\cos\alpha + \frac{\partial u}{\partial y}\cos\beta + \frac{\partial u}{\partial z}\rho\cos\gamma + \varepsilon..$$

$\rho \to 0$ bolanda $\frac{\varepsilon}{\rho} \to 0.$

$$\frac{u(P_1) - u(P)}{\rho} = \frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\cos\beta + \frac{\partial u}{\partial z}\cos\gamma + \frac{\varepsilon}{\rho}.$$

$\rho \to 0$ bolanda predele geçip alarys:

$$\frac{\partial u}{\partial \lambda} = \lim_{\rho\to 0}\frac{U(P_1) - U(P)}{\rho} = \frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\cos\beta + \frac{\partial u}{\partial z}\cos\gamma.$$

Teorema subut edildi.

1-nji mysal. $u = x^2 + y^2 + 2xy$ skalýar meýdanyň Ox ok bilen $\alpha = 60^\circ$ burçy emele getirýän ugur boýunça önümini tapmaly.

Çözülişi. $u = f(x,y)$ endigan meýdan bolsa, onda $M_0(x_0,y_0)$ nokatda $\boldsymbol{l}$ ugur boýunça önüm

$$\left.\frac{\partial u}{\partial l}\right|_{M_0} = \left.\frac{\partial u}{\partial x}\right|_{M_0}\cos\alpha + \left.\frac{\partial u}{\partial y}\right|_{M_0}\cos\beta \quad \beta = 90^\circ - \alpha, \cos\beta = \sin\alpha$$

ýa-da

$$\left.\frac{\partial u}{\partial l}\right|_{M_0} = \left.\frac{\partial u}{\partial x}\right|_{M_0}\cos\alpha + \left.\frac{\partial u}{\partial y}\right|_{M_0}\sin\alpha$$

deňlik bilen tapylýar. Bu ýerde α burç käbir $\boldsymbol{l}$ wektor bilen Ox okuň aralygyndaky burçdur.

$$\cos 60^\circ = \frac{1}{2}, \sin 60^\circ = \frac{\sqrt{3}}{2}, \frac{\partial u}{\partial x} = 2x + 2y, \frac{\partial u}{\partial y} = 2y + 2x..$$

$$\frac{\partial u}{\partial l} = \frac{\partial u}{\partial x}\cos 60^\circ + \frac{\partial u}{\partial y}\sin 60^\circ = (2x + 2y)\frac{1}{2} + (2y + 2x)\frac{\sqrt{3}}{2} =$$
$$= (x + y)(1 + \sqrt{3}).$$

2-nji mysal. u=xyz skalýar meýdanyň $P(5,1,2)$ nokatdaky önümini, bu nokatdan Q(7,–1,3) nokada görä ugur boýunça tapmaly.

Çözülişi. $\overrightarrow{PQ}\{2, -2, 1\}$ wektoryň uzynlygy

$|\overrightarrow{PQ}| = \sqrt{2^2 + (-2)^2 + 1^2} = 3$. Şeýle hem, $\overrightarrow{PQ} = 2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$.

Ugrukdyryjy kosinuslary

$$\cos\alpha\frac{2}{3}, \cos\beta = -\frac{2}{3}, \cos\gamma = \frac{1}{3}.$$

u=xyz funksiýanyň hususy önümleriniň P(5,1,2) nokatdaky bahalaryny tapalyň:

$$\left.\frac{\partial u}{\partial x}\right|_P = yz\big|_P = 2, \left.\frac{\partial u}{\partial y}\right|_P = xz\big|_P = 10, \left.\frac{\partial u}{\partial z}\right|_P = xy\big|_P = 5.$$

Şunlukda,

$$\frac{\partial u}{\partial \lambda} = 2 \cdot \frac{2}{3} + 10 \cdot \left(-\frac{2}{3}\right) + 5 \cdot \frac{1}{3} = -\frac{11}{3}.$$

Berlen ugur boýunça funksiýanyň kemelýändigini aýyrmak alamaty görkezýär.

3-nji mysal. $u = xz^2 + 2zy$ skalýar meýdanyň

$$\begin{cases} x = 1 + \cos t \\ y = \sin t - 1 \\ z = 2 \end{cases}$$

töweregiň $M(1,0,2)$ nokadynda geçirilen galtaşýan göni çyzygyň ugry boýunça önümini tapmaly.

Çözülişi. Berlen töweregiň deňlemesini wektor görnüşde ýazalyň:

$$\boldsymbol{r}(t) = (1 + \cos t)\boldsymbol{i} + (\sin t - 1)\boldsymbol{j} + \boldsymbol{k}.$$

Bu töwerege erkin M_0 nokatda galtaşýan $\boldsymbol{\tau}$ wektory tapalyň:

$$\boldsymbol{\tau} = \frac{d\boldsymbol{r}}{dt} = -\boldsymbol{i}\sin t + \boldsymbol{j}\cos t.$$

Berlen $M(1,0,2)$ nokat xOy tekizligiň l oktandynda ýerleşýänligi sebäpli $t = \frac{\pi}{2}$ we $\boldsymbol{\tau}\big|_M = -\boldsymbol{i}\sin\frac{\pi}{2} + \boldsymbol{j}\cos\frac{\pi}{2} = -\boldsymbol{i}$.

Onda ugrukdyryjy kosinuslar aşakdaky görnüşde bolarlar: $\cos\alpha = -1, \cos\beta = 0, \cos\gamma = 0$. Skalýar meýdanyň hususy önümleriniň $M(1,0,2)$ nokatdaky bahalaryny tapalyň:

$$\frac{\partial u}{\partial x}\bigg|_M = z^2\big|_M = 4, \frac{\partial u}{\partial y}\bigg|_M = 2x\big|_M = 4, \frac{\partial u}{\partial z}\bigg|_M = (2xz + 2y)\big|_M.$$

Onda

$$\frac{\partial u}{\partial l}\bigg|_M = \frac{\partial u}{\partial \tau}\bigg|_M = 4(-1) + 4 \cdot 0 + 4 \cdot 0 = -4.$$

4-nji mysal. $u = In(x^2 + y^2)$ skalýar meýdanyň $y^2 = 4x$ parabolanyň $M(1,2)$ nokadynda geçirilen galtaşýan göni çyzygyň ugry boýunça önümini tapmaly.

Çözülişi. $y^2 = 4x$ parabolanyň $M(1,2)$ nokadyndaky ugruny bu nokatda geçirilen galtaşýanyň ugry bilen gabat gelýär diýip kabul edýäris. Goý, M nokatda geçirilen galtaşýan Ox ok bilen α burçy emele getirsin. Onda $y = 2\sqrt{x}, y' = \dfrac{1}{\sqrt{x}}, \operatorname{tg}\alpha = y'\big|_{x-1} = 1$. Ugrukdyryjy kosinuslary tapalyň:

$$\cos\alpha = \frac{1}{\sqrt{1+\operatorname{tg}^2\alpha}} = \frac{1}{\sqrt{2}}, \cos\beta = \sin\alpha = \frac{\operatorname{tg}\alpha}{\sqrt{1+\operatorname{tg}^2}} = \frac{1}{\sqrt{2}}.$$

Berlen $u = \ln(x^2 + y^2)$ funksiýanyň önümleriniň $M(1,2)$ nokatdaky bahalaryny tapalyň:

$$\frac{\partial u}{\partial x}\bigg|_M = \frac{2x}{x^2+y^2}\bigg|_M = \frac{2}{1+2^2} = \frac{2}{5}, \frac{\partial u}{\partial y}\bigg|_M = \frac{2y}{x^2+y^2}\bigg|_M = \frac{4}{1+2^2}.$$

Şunlukda,

$$\frac{\partial u}{\partial l} = \frac{2}{5}\frac{1}{\sqrt{2}} + \frac{4}{5}\frac{1}{\sqrt{2}} = \frac{6}{5\sqrt{2}} = \frac{3\sqrt{2}}{5}.$$

11. $u = 2x^2 - 3y^2$ skalýar meýdanyň $P(1,0)$ nokatda Ox ok bilen $120°$ burcy emele getirýän ugur boýunça önümini tapmaly.

12. $u = x^2 - y^2$ skalýar meýdanyň $P(1,1)$ nokatda Ox ok bilen $60°$ burçy emele getirýän ugur boýunça önümini tapmaly.

13. $u=xyz$ skalýar meýdanyň $M(1,1,1)$ nokatdaky önümini $\boldsymbol{l}\{\cos\alpha, \cos\beta, \cos\gamma\}$ ugur boýunça tapmaly.

14. $u = \operatorname{arctg}xy$ skalýar meýdanyň önümini, $y = x^2$ parabola degişli $M(1,1)$ nokatda, şu egriniň ugry boýunça tapmaly.

15. $u = x^2 - 2xz + y^2$ skalýar meýdanyň $P(1,2,-1)$ nokatdaky önümini P nokatdan $M(2,4,-3)$ nokada görä ugur boýunça tapmaly.

16. $y = \frac{x^2}{2}$ parabolanyň $P\left(1,\frac{1}{2}\right)$ nokadynda parabola geçirilen galtaşýanyň ugry boýunça $u = In(x + 2y)$ skalýar meýdanyň önümini tapmaly.

17. Eger $r = \sqrt{x^2 + y^2 + z^2}$ bolsa, $u = \frac{1}{r}$ skalýar meýdanyň $\boldsymbol{l}\{\cos\alpha, \cos\beta, \cos\gamma\}$ ugur boýunça önümini tapmaly.

Berlen funksiýanyň $M(x,y,z)$ nokatdaky önümini bu nokatdan $N(x,y,z)$ nokada görä ugur boýunça tapmaly.

18. $u = \sqrt{x^2 + y^2 + z^2}$, $M(1,1,1)$, $N(3,2,1)$.

19. $u = x^2y + xz^2 - 2$, $M(1,1,-1)$, $N(2,-1,3)$.

20. $u = \frac{x}{y} - \frac{y}{x}$, $M(1,1)$, $N(4,5)$.

21. $u = \ln(xy + yz + xz)$ skalýar meýdanyň $x=\cos t$, $y=\sin t$, $z=1$ töweregiň $M(0,1,1)$ nokadynda geçirilen galtaşýan göni çyzygyň ugry boýunça önümini tapmaly.

22. $u = x^2 + y^2 + z^2$ skalýar meýdanyň $x=R\cos t$, $y=R\sin t$, $z=at$ hyrly aýlaw çyzygyň $t = \frac{\pi}{2}$ degişli bolan $M(x,y,z)$ nokadynda geçirilen galtaşýan göni çyzygyň ugry boýunça önümini tapmaly.

§ 3. Gradiýent

Berlen ugur boýunça önüm almaklygyň formulasyna garalyň:

$$\frac{\partial u}{\partial \lambda}=\frac{\partial u}{\partial x}\cos\alpha+\frac{\partial u}{\partial y}\cos\beta+\frac{\partial u}{\partial z}\cos\gamma.$$

Deňligiň sag bölegininiň her bir goşulyjysynyň ikinji köpeldijisi berlen λ şöhläniň ugry boýunça ugrukdyrylan, $\boldsymbol{e}_{\lambda}$ birlik wektoryň proýeksiýalarydyr:

$$\boldsymbol{e}_{\lambda}(\cos\alpha,\cos\beta,\cos\gamma).$$

Indi koordinata oklara proýeksiýalary $\frac{\partial u_x}{\partial x},\frac{\partial u_y}{\partial y},\frac{\partial u_z}{\partial y}$ hususy önümleriň berlen $P(x,y,z)$ nokatdaky bahalaryna deň bolan wektory alalyň. Şeýle wektora $u(x,y,z)$ funksiýanyň gradiýenti diýilýär we grad$u$ ýa-da ∇u belgi bilen belgilenýär. Şunlukda, biz aşakdaky kesgitlemäni aldyk.

Kesgitleme. $u=(x,y,z)$ funksiýanyň (x,y,z) nokatdaky gradiýent wektory diýip grad$u$ bilen belgilenýän we

$$\text{grad}\, u=\frac{\partial u_x}{\partial x}i+\frac{\partial u_y}{\partial y}j+\frac{\partial u_z}{\partial z}k$$

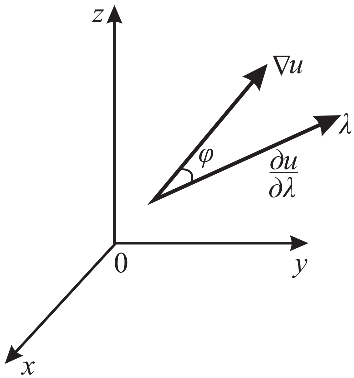
formula bilen kesgitlenýän wektora aýdylýar.

Görnüşi ýaly gradiýentiň proýeksiýalary $P(x,y,z)$ nokadyň alnyşyna baglydyr we bu nokadyň koordinatalarynyň üýtgemegi bilen gradiýent hem üýtgeýändir. Şunlukda, $u(x,y,z)$ funksiýa bilen kesgitlenýän skalýar meýdanyň her bir nokadyna bir wektor degişlidir, ol şol funksiýanyň gradiýentidir. $u=az+by+cz+d$ çyzykly funksiýanyň gradiýenti $\text{grad}u=a\boldsymbol{i}+b\boldsymbol{j}+c\boldsymbol{k}$ hemişelik wektordyr.

Skalýar meýdanyň $\boldsymbol{\lambda}$ ugur boýunça önümini aşakdaky görnüşde ýazyp bileris: $\frac{\partial \boldsymbol{u}}{\partial\lambda}=\text{grad}u\cdot e_{\lambda}$. Şeýlelikde, skalýar meýdanyň $\boldsymbol{\lambda}$ ugur boýunça önümi, u funksiýanyň gradiýentiniň, birlik wektora skalýar köpeltmek hasylyna deňdir:

$$\frac{\partial \boldsymbol{u}}{\partial \lambda} = \left| \operatorname{grad} u \right| \cdot \cos\varphi.$$

Bu ýerde φ, grad$u$ bilen λ şöhläniň aralygyndaky burçdur. Şu ýerden görnüşi ýaly ugur boýunça önüm iň uly bahany $\cos\varphi = 1, \varphi = 0$ bolanda alýar we ol $\left|\operatorname{grad} u\right|$ deňdir.



2-nji surat

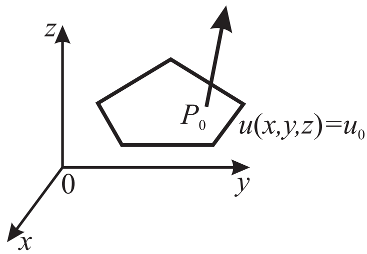
Aşakdaky teorema funksiýanyň gradiýentiniň ugry bilen skalýar meýdanyň dereje üstüni baglanyşdyrýar.

Teorema. $u(x,y,z)$ funksiýanyň her bir nokatdaky gradiýentiniň ugry, şu nokatda skalýar meýdanyň dereje üstüne geçirilen normalyň ugry bilen gabat gelýär.

Subudy. Erkin $P_0(x_0, y_0, z_0)$ nokady alalyň. Şu nokat arkaly geçýän dereje üstüň deňlemesi $u = (x, y, z) = u_0, u_0 = u(x_0, y_0, z_0)$ görnüşdedir.

Bu üste $P_0(x_0, y_0, z_0)$ nokatda geçirilen normalyň deňlemesini ýazalyň:

$$\frac{x - x_0}{\left(\frac{\partial u}{\partial x}\right)_0} = \frac{y - y_0}{\left(\frac{\partial u}{\partial y}\right)_0} = \frac{z - z_0}{\left(\frac{\partial u}{\partial z}\right)}.$$



3-nji surat

Bu ýerden görnüşi ýaly proýeksiyalary $\left(\frac{\partial \boldsymbol{u}}{\partial x}\right)_0, \left(\frac{\partial \boldsymbol{u}}{\partial y}\right)_0, \left(\frac{\partial \boldsymbol{u}}{\partial z}\right)_0$ bolan ugrukdyryjy normal wektor, $P_0(x_0, y_0, z_0)$ nokatda $u = (x, y, z)$ funksiýanyň gradiýentidir.

Teorema subut edildi.

Şunlukda, gradiýent berlen nokatda dereje üste geçirilen galtaşýan tekizligiň her bir nokadyna perpendikulýardyr we onuň bu tekizlige proýeksiýasy nola deňdir.

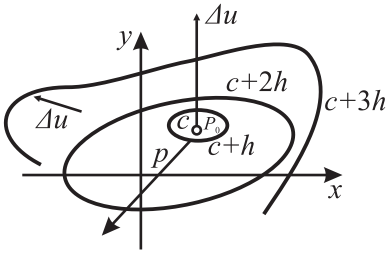
Ol bolsa dereje üste geçirilen galtaşýan tekizligiň her bir nokadynda, erkin ugur boýunça alnan önümiň nola deňdigini görkezýär.

$u=u(x,y)$ tekiz meýdanda $\text{grad}u = \frac{\partial u}{\partial x}\boldsymbol{i} + \frac{\partial u}{\partial y}\boldsymbol{j}$ gradiýent Oxy tekizlikde ýatýar we ol dereje çyzyga perpendikulýardyr.

Eger tekiz meýdanda köp sanly dereje çyzyklaryň toplumy gurlan bolsa, onda çyzgynyň kömegi bilen, ýeterlik takmynlykda, gradiýentiň modulyny we ugruny kesgitläp bolýar. Gradiýentiň ugry dereje çyzyga perpendikulýardyr. Bu ugur boýunça önüm, ýeterlik kiçi h üçin takmynan

$$\frac{\partial u}{\partial \lambda} \approx \frac{u(P) - u(P_0)}{P_0P} = \frac{h}{P_0P}$$

ululyga deňdir. Bu ýerde P_0 nokat $u = (x, y) = c$ dereje üstüň nokady, P nokat bolsa $u(x, y) = c + h$ dereje üstüň nokadydyr. h ululyk bellidir. P_0P kesimiň uzynlygyny bolsa, dereje çyzyklaryň normallarynyň aralygyndaky uzaklyk hökmünde ölçenip bilner.



4-nji surat

Gradiýentiň ugry boýunça alnan önüm onuň modulyna deňdir. Şonuň üçin

$$\left|\operatorname{grad}u\right| \approx \frac{h}{P_0P}.$$

§ 4. Gradiýentiň esasy häsiýetleri

Indi gradiýenti hasaplamagy ýeňilleşdirýän onuň birnäçe häsiýetlerine garalyň.

1. $\operatorname{grad}(u_1 + u_2) = \operatorname{grad}u_1 + \operatorname{grad}u_2$.

$$\operatorname{grad}(u_1 + u_2) = \frac{\partial(u_1 + u_2)}{\partial x}\boldsymbol{i} + \frac{\partial(u_1 + u_2)}{\partial y}\boldsymbol{j} + \frac{\partial(u_1 + u_2)}{\partial z}\boldsymbol{k} =$$

$$= \frac{\partial u_1}{\partial x}\boldsymbol{i} + \frac{\partial u_2}{\partial x}\boldsymbol{i} + \frac{\partial u_1}{\partial y}\boldsymbol{j} + \frac{\partial u_2}{\partial y}\boldsymbol{j} + \frac{\partial u_1}{\partial z}\boldsymbol{k} + \frac{\partial u_2}{\partial z}\boldsymbol{k} =$$

$$= \left(\frac{\partial u_1}{\partial x}\boldsymbol{i} + \frac{\partial u_1}{\partial y}\boldsymbol{j} + \frac{\partial u_1}{\partial z}\boldsymbol{k}\right) + \left(\frac{\partial u_2}{\partial x}\boldsymbol{i} + \frac{\partial u_2}{\partial y}\boldsymbol{j} + \frac{\partial u_2}{\partial z}\boldsymbol{k}\right) = \operatorname{grad}u_1 + \operatorname{grad}u_2.$$

2. $\operatorname{grad}c \cdot u = c \cdot \operatorname{grad}u$

$$\operatorname{grad}cu = \frac{\partial cu}{\partial x}\boldsymbol{i} + \frac{\partial cu}{\partial y}\boldsymbol{j} + \frac{\partial cu}{\partial z}\boldsymbol{k} = c\left(\frac{\partial u}{\partial x}\boldsymbol{i} + \frac{\partial u}{\partial y}\boldsymbol{j} + \frac{\partial u}{\partial z}\boldsymbol{k}\right) = c\operatorname{grad}u.$$

3. $\operatorname{grad}u_1u_2 = u_2\operatorname{grad}u_1 + u_1\operatorname{grad}u_2$.

$$\operatorname{grad} u_1 u_2 = \frac{\partial(u_1 u_2)}{\partial x}\boldsymbol{i} + \frac{\partial(u_1 u_2)}{\partial y}\boldsymbol{j} + \frac{\partial(u_1 u_2)}{\partial z}\boldsymbol{k} =$$

$$= u_2\left(\frac{\partial u_1}{\partial x}\boldsymbol{i} + \frac{\partial u_1}{\partial y}\boldsymbol{j} + \frac{\partial u_1}{\partial z}\boldsymbol{k}\right) + u_1\left(\frac{\partial u_2}{\partial x}\boldsymbol{i} + \frac{\partial u_2}{\partial y}\boldsymbol{j} + \frac{\partial u_2}{\partial z}\boldsymbol{k}\right) =$$
$$= u_2 \operatorname{grad} u_1 + u_1 \operatorname{grad} u_2.$$

4. $\operatorname{grad} f(u) = f'(u)\operatorname{grad} u.$

$$\operatorname{grad} f(u) = \frac{\partial}{\partial x}[f(u)]\boldsymbol{i} + \frac{\partial}{\partial y}[f(u)]\boldsymbol{j} + \frac{\partial}{\partial z}[f(u)]\boldsymbol{k} = f'(u)\frac{\partial u}{\partial x}\boldsymbol{i} +$$
$$+ f'(u)\frac{\partial u}{\partial y}\boldsymbol{j} + f'(u)\frac{\partial u}{\partial z}\boldsymbol{k} = f'(u)\operatorname{grad} u.$$

1-nji mysal. $u = \sqrt{2xy + y^2}$ skalýar meýdanyň $M(3,2)$ nokatdaky gradiýentini tapmaly.

Çözülişi. $u = \sqrt{2xy + y^2}$ funksiýanyň hususy önümleriniň $M(3,2)$ nokatdaky bahalaryny tapalyň:

$$\left.\frac{\partial u}{\partial x}\right|_M = \left.\frac{y}{\sqrt{2xy + y^2}}\right|_M = \frac{2}{\sqrt{12 + 2^2}} = \frac{1}{2};$$

$$\left.\frac{\partial u}{\partial y}\right|_M = \left.\frac{x + y}{\sqrt{2xy + y^2}}\right|_M = \frac{5}{4}.$$

Onda,

$$\operatorname{grad} u = \frac{1}{2}\boldsymbol{i} + \frac{5}{4}\boldsymbol{j}.$$

2-nji mysal. Eger $O(0,0,0)$ nokat koordinatalar başlangyjy, $M(x,y.z)$ giňişligiň erkin nokady bolsa,

$$r = \sqrt{x^2 + y^2 + z^2}$$

uzaklygyň gradiýentini tapmaly.

Çözülişi.

$$\operatorname{grad} u = \frac{\partial r}{\partial x}\boldsymbol{i} + \frac{\partial r}{\partial y}\boldsymbol{j} + \frac{\partial r}{\partial z}\boldsymbol{k} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}\boldsymbol{i} + \frac{y}{\sqrt{x^2 + y^2 + z^2}}\boldsymbol{j}$$
$$+ \frac{z}{\sqrt{x^2 + y^2 + z^2}}\boldsymbol{k} = \frac{x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}}{r}.$$

3-nji mysal. $u = x^2y + y^2z + z^2x$ skalýar meýdanyň $M(1,0,0)$ nokatdaky iň uly üýtgemesini tapmaly.

Çözülişi.

$$\frac{\partial u}{\partial x} = 2xy + z^2, \frac{\partial u}{\partial y} = x^2 + 2yz, \frac{\partial u}{\partial z} = y^2 + 2zx,$$

$$\operatorname{grad} u = (2xy + z^2)\boldsymbol{i} + (x^2 + 2yz)\boldsymbol{j} + (y^2 + 2zx)\boldsymbol{k},$$

$$\operatorname{grad} u\big|_M = \boldsymbol{j}, \left(\frac{\partial u}{\partial l}\right)\Bigg|_{\max} = \left|\operatorname{grad} u\right| = 1.$$

4-nji mysal. $u = xy - z^2$ skalýar meýdanyň gradiýentini we gradiýentiň $M(-9,12,10)$ nokatdaky ugruny kesgitlemeli.

Çözülişi.Gradiýentiň kesgitlemesini ulanyp, alarys:

$$\operatorname{grad} u(M) = \frac{\partial u}{\partial x}\bigg|_M \boldsymbol{i} + \frac{\partial u}{\partial y}\bigg|_M \boldsymbol{j} + \frac{\partial u}{\partial z}\bigg|_M = y\big|_M \boldsymbol{i} + x\big|_M - 2z\big|_M = 12\boldsymbol{i}$$

$$\left|\operatorname{grad} u(M)\right| = \sqrt{12^2 + (-9)^2 + (-20)^2} = 25.$$

Gradiýentiň ugry

$$\boldsymbol{l}(M) = \frac{\operatorname{grad} u(M)}{\left|\operatorname{grad} u(M)\right|}.$$

wektor bilen tapylýar. Onda

$$\boldsymbol{l}(M) = \frac{12\boldsymbol{i} - 9\boldsymbol{j} - 20\boldsymbol{k}}{25} = \frac{12}{25}\boldsymbol{i} - \frac{9}{25}\boldsymbol{j} - \frac{4}{5}\boldsymbol{k}$$

$$\cos\alpha = \frac{12}{25}, \cos\beta = -\frac{9}{25}, \cos\gamma = -\frac{4}{5}.$$

5-nji mysal. $u = \dfrac{x}{x^2 + y^2 + z^2}$ skalýar meýdanyň $A(1,2,2)$ we $B(-3,1,0)$ nokatlardaky gradiýentleriniň aralygyndaky φ burçy tapmaly.

Çözülişi. Gradiýentleriň aralygyndaky φ burçy aşakdaky formula bilen taparys

$$\cos\varphi = \frac{(\mathrm{grad}u(A), \mathrm{grad}u(B))}{|\mathrm{grad}u(A)\|\mathrm{grad}u(B)|}$$

$$\frac{\partial u}{\partial x} = \frac{x^2+y^2+z^2-2x^2}{(x^2+y^2+z^2)^2} = \frac{1}{r^2} - \frac{2x^2}{r^4},\ r^2 = x^2+y^2+z^2,,$$

$$\frac{\partial u}{\partial y} = -\frac{2xy}{r^4}, \frac{\partial u}{\partial z} = -\frac{2zx}{r^4}, r(A) = \sqrt{1+2^2+2^2} = 3,$$

$$r(B) = \sqrt{(-3)^2+1^2} = \sqrt{10}, \frac{\partial u(A)}{\partial x} = \frac{1}{9} - \frac{2}{81} = \frac{7}{81},$$

$$\frac{\partial u(A)}{\partial y} = -\frac{4}{81}, \frac{\partial u(A)}{\partial z} = -\frac{4}{81}, \frac{\partial u(B)}{\partial x} = -\frac{2}{25},$$

$$\frac{\partial u(B)}{\partial y} = \frac{3}{50}, \frac{\partial u(B)}{\partial z} = 0.$$

$$\mathrm{grad}u(A) = \frac{7}{81}\boldsymbol{i} - \frac{4}{81}\boldsymbol{j} - \frac{4}{81}\boldsymbol{k} \quad \mathrm{grad}u(B) = -\frac{2}{25}\boldsymbol{i} + \frac{3}{50}\boldsymbol{j}$$

$$|\mathrm{grad}u(A)| = \frac{1}{81}\sqrt{49+16+16} = \frac{1}{9}$$

$$|\mathrm{grad}u(B)| = \sqrt{\frac{4}{625} + \frac{9}{2500}} = \frac{1}{10}$$

$$(\mathrm{grad}u(A), \mathrm{grad}u(B)) = -\frac{14}{2025} - \frac{12}{4050} = -\frac{4}{405}$$

$$\cos\varphi = -\frac{4}{405} : \frac{1}{90} = -\frac{4}{9}.$$

6-njy mysal. $u = x^3 + y^3 + z^3 - 3xyz$ skalýar meýdanyň gradiýenti $Oxyz$ giňişligiň haýsy nokatlarynda, a) Oz oka perpendikulýar, b) Oz oka parallel, ç) nola deň?

Çözülişi. Gradiýentiň kesgitlemesiniň esasynda, ýazyp bileris:

$\mathrm{grad}u(x,y,z) = \boldsymbol{i}^3(x^2 - yz) + \boldsymbol{j}^3(y^2 - xz) + \boldsymbol{k}^3(z^2 - xy)$.

a) halda $z^2 - xy = 0$ deňlik ýerine ýetýär, şonuň üçin $z^2 = xy$;

b) halda grad$u(x,y,z)$ wektor k wektora kolleniardyr. Onda gözleýän köplügimizde bir wagtda, $x^2 - yz = 0, y^2 - xz = 0$ deňlikler ýerine ýetýärler. Bu deňliklerden $x^3 - y^3 = 0$ ýa-da $(x-y)(x^2 + xy + y^2) = 0$ deňligi alýarys. Bu ýerden $x=y$, ikinji deňlik bolsa $x=y=0$ bolanda ýerine ýetýär. $x=y$ deňligi, deňlikleriň erkin birisine goýup $x=y=z$ bolýandygyny görýäris.

ç) halda bolsa, $x^2 - yz = 0, y^2 - xz = 0, z^2 - xy = 0$, deňlikler ýerine ýetýär. Bu bolsa $x=y=z$ bolanda dogrudyr.

23. $u = x - 2y + 3z$ skalýar meýdanyň gradiýentini tapmaly.

24. $u = \ln(x^2 + y^2 + z^2)$ skalýar meýdanyň $M(1,1,-1)$ nokatda gradiýentini tapmaly.

25. $u = x^2 + 2y^2 + 3z^2 + xy + 3x - 2y - 6z$ skalýar meýdanyň $O(0,0,0)$ nokatda gradiýentini tapmaly.

26. $u = x^y$ skalýar meýdanyň $M(2,2,4)$ nokatdaky iň uly üýtgemesini tapmaly.

27. $u = (x-y)(y-z)(z-x)$ skalýar meýdanyň $M(1.2.3)$, $N(3,2,1)$, $P(2,3,4)$ nokatlardaky gradiýentini tapmaly.

28. $u = (x-1)(y-2)(z-3)$ skalýar meýdanyň $P(2,3,4)$ nokatda gradiýentini tapmaly.

29. $u = ze^{x^2+y^2+z^2}$ skalýar meýdanyň $O(0,0,0)$ nokatda gradiýentini tapmaly.

30. $u = x^2 + 2y^2 - z^2 - 5$ skalýar meýdanyň $P(2,-1,1)$ nokatda gradiýentini tapmaly.

31. $u_1 = \sqrt{x^2 + y^2}$ we $u_2 = x + y + 2\sqrt{xy}$ skalýar meýdanlaryň M(1,1) nokatdaky gradiýentleriniň aralygyndaky φ burçy tapmaly.

32. $u = \mathrm{arctg}\dfrac{x}{y}$ skalýar meýdanyň $M(1,1)$ we N(–1,–1) nokatlardaky gradiýentleriniň aralygyndaky φ burçy tapmaly.

33. $u = (x+y)e^{x+y}$skalýar meýdanyň $M(0,0)$ we $N(1,1)$ nokatlardaky gradiýentleriniň aralygyndaky φ burçy tapmaly.

34. $u_1 = \sqrt{x^2 + y^2 + z^2}$ we $u_2 = \ln(x^2 + y^2 + z^2)$ skalýar meýdanlaryň $M(0,0,1)$ nokatdaky gradiýentleriniň aralygyndaky φ burçy tapmaly.

35. $u = x^2 + y^2 + z^2$ skalýar meýdanyň $A(\varepsilon, 0, 0)$ we $B(0, \varepsilon, 0)$ nokatlardaky gradiýentleriniň aralygyndaky φ burçy tapmaly.

36. $u(M) = xyz$ skalýar meýdanyň $M(2,1,-1)$ nokatdaky ugruny we üýtgemesiniň iň uly ululygyny kesgitlemeli.

37. $u = x^2 y - 5y^3$ skalýar meýdanyň $P(2,1)$ nokatda üýtgemeginiň uň uly ululygyny kesgitlemeli.

38. $u = \ln\frac{1}{r}, r = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$ skalýar meýdan bolsa, giňişligiň haýsy nokatlarynda $|\mathrm{grad}\, u| = 1$ şert ýerine ýetýär?

II BÖLÜM
WEKTORLY MEÝDANLAR

§ 1. Wektorly meýdanlaryň kesgitlenilişi

Wektorly meýdanyň kesgitlenilişi seredip geçen skalýar meýdanynyň kesgitlenilişine meňzeşdir.

Kesgitleme. Eger D ýaýlanyň her bir P nokadyna wektor degişli bolsa, onda bu ýaýlada wektorly meýdan berlen diýilýär.

Kesgitlemeden görnüşi ýaly, wektorly meýdan tutuş giňişlik ýa-da onuň belli bir bölegi hem bolup biler. Şeýle hem her bir P nokada $\boldsymbol{A}(P)$ wektor degişli bolsa, onda bu nokatlaryň köplügi wektorly meýdany emele getirýär. $\boldsymbol{A}(P)$ wektoryň koordinata oklara bolan proýeksiýalaryny A_x, A_y, A_z bilen, P nokadyň koordinatalaryny bolsa x, y, z bilen belgilesek, $\boldsymbol{A}(P)$ wektory aşakdaky görnüşde ýazyp bileris:

$$\boldsymbol{A}(P) = A_x(x,y,z)\boldsymbol{i} + A_y(x,y,z)\boldsymbol{j} + A_z(x,y,z)\boldsymbol{k},$$

bu ýerde A_x, A_y, A_z funksiýalar üznüksizdirler we üznüksiz önümlere eýedirler.

Wektorly meýdanyň mysallary bolup güýçleriň meýdany, tizlikleriň meýdany, elektrik we magnit meýdanlary hem-de şuňa meňzeş mysallar hyzmat edýärler. Eger wektorly meýdanyň $\boldsymbol{A}(P)$ wektory diňe P nokada bagly bolup, t wagta bolsa bagly bolmasa, onda ol meýdan stasionar meýdandyr, şeýle hem $\boldsymbol{A}(P)$ wektor P nokada hem-de t wagta bagly bolsa, onda ol meýdan stasionar meýdan däldir.

Wektorly meýdanyň hususy hallaryna garalyň.

1. Biratly meýdan. Eger $\boldsymbol{A}(P)$ hemişelik wektor bolsa, ýagny $\boldsymbol{A}(P)$ wektoryň koordinata oklaryna bolan A_x, A_y, A_z proýeksiýalary hemişelik ululyklar bolsa, onda bu wektorly meýdan biratly meýdandyr.

2. Tekiz meýdan. Eger wektorly meýdanyň $\boldsymbol{A}(P)$ wektorynyň A_x, A_y, A_z proýeksiýalary x,y,z ululyklaryň haýsy hem bolsa birine bagly däl bolup, onuň bir proýeksiýasy nola deň bolsa, onda ol meýdan tekiz meýdandyr.

Mysal üçin,

$\boldsymbol{A}(P) = A_x(x,y)\boldsymbol{i} + A(x,y)\boldsymbol{j}$

wektorlar bilen emele gelýän meýdanlar tekiz meýdanlardyr. Tekiz meýdanlar gidrodinamikada suwuklyklaryň tekiz akymy öwrenilen mahalynda duş gelýärler. Bu akymlarda suwuklyklaryň ähli bölejikleri haýsy hem bolsa bir tekizlige parallel bolup hereket edýärler we tekizlige perpendikulýar bolan gönüdäki bölejikleriň hereketleri birmeňzeşdirler.

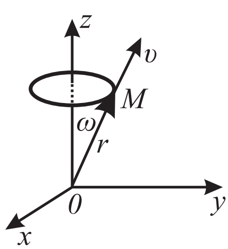
Tekiz meýdanyň zerur fiziki mysalynyň birisine garap geçeliň.

1-nji mysal. Eger, gaty jisim hemişelik ω burçlaýyn tizlik bilen hereket edýän bolsa, jisimiň çyzyklaýyn tizlikleriiniň meýdanyny tapmaly.

Çözülişi. Kinematikadan belli bolşy ýaly, islendik M nokadyň $\boldsymbol{\vartheta}$ çyzyk tizligi, onuň $\boldsymbol{\omega}$ burçlaýyn tizliginiň, nokadyň $\boldsymbol{r}$ radius-wektoryna, wektorlaýyn köpeltmek hasylyna deňdir:

$$\boldsymbol{\vartheta}=[\boldsymbol{\omega}.\boldsymbol{r}];$$

$$\boldsymbol{\vartheta}=[\boldsymbol{\omega}\times\boldsymbol{r}]$$



5-nji surat

Bu ýerde $\boldsymbol{\omega} = \omega\boldsymbol{k}, \boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$. Onda

$$\vartheta = [\omega \times \boldsymbol{r}] = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = x\boldsymbol{j}\omega - \boldsymbol{i}y\omega.$$

Şunlukda, $\vartheta_x = - y\omega, \vartheta_y = x\omega, \vartheta_z = 0.$

Diýmek, bu meýdan tekiz meýdandyr.

2-nji mysal. Goý, koordinatalar başlangyjynda m massaly O nokat ýerleşen bolsun. Onda $M(x,y,z)$ nokatda ýerleşen m_1 massaly nokady, O nokat

$$\boldsymbol{F} = - \gamma \frac{m}{r^3}(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k})$$

güýç bilen özüne dartýar. Giňişligiň her bir nokadynda kesgitlenen, bu dartylma güýçler, wektorly meýdany, ýagny nokatlaýyn m massaly dartylma güýçleriň meýdanyny emele getirýär.

38 a. Material nokat ýeriň töwereginde hereket edip, $A(x_0, y_0, z_0)$ nokatdan $B(x_1, y_1 z_1)$ nokada geçipdir. Ýeriň dartyş güýjüniň şu geçişde bitiren işini kesgitlemeli.

§ 2. Wektorly çyzyklar

Kesgitleme. Eger wektorly meýdandaky çyzygyň her bir hokadyna geçirilen galtaşýanyň ugry şol nokatdaky wektoryň ugry bilen gabat gelse, bu çyzyga wektorly meýdanyň wektorly çyzygy diýilýär.

Wektorly çyzyklaryň takyk hasaplamalarda anyk fiziki manysy bardyr.

Eger biz akýan suwuklygyň tizlikleriniň meýdanyna garasak, wektorly çyzyklar jisimiň bölejikleriniň akymynyň çyzyklarydyr.

Elektrik meýdanynda wektorly çyzyklar bu meýdanyň güýç çyzyklarydyr. Mysal üçin, nokatlaýyn zarýadlaryň meýdanynda wektorly çyzyklar, zarýadlardan çykýan şöhlelerdir. Magnit meýdanynda wektorly çyzyklary demirgazyk polýusdan başlap günorta polýusda gutarýan çyzyklardyr.

Güýç çyzyklarynyň elektrik, magnit we elektromagnit meýdanynda ýerleşişi fizikanyň meseleleri öwrenilen mahalynda esasy orny eýeleýär.

Wektorly çyzyklaryň deňlemesini getirip çykaralyň.

Goy, wektorly meýdan

$$\boldsymbol{A}(P) = A_x(x,y,z)\boldsymbol{i} + A_y(x,y,z)\boldsymbol{j} + A_z(x,y,z)\boldsymbol{k}$$

funksiýa bilen kesgitlenen bolsun. Eger wektorly çyzyklar özüniň parametrli deňlemesi bilen berlen bolsa:

$$x = x(t), y = y(t), z = z(t),$$

onda oňa galtaşýan wektoryň ugrukdyryjylarynyň koordinata oklara bolan proýeksiýalary $x'(t), y'(t), z'(t)$ önümlere ýa-da *dx, dy*, *dz* differensiallara proporsionaldyrlar:

$$\frac{x-x_0}{x'(t)} = \frac{y-y_0}{y'(t)} = \frac{z-z_0}{z'(t)}, \frac{dx}{x'(t)} = \frac{dy}{y'(t)} = \frac{dz}{z'(t)}.$$

$\boldsymbol{A}(P)$ wektor bilen, wektorly çyzyga galtaşýan wektoryň parallellik şertini ýazalyň:

$$\frac{dx}{A_x} = \frac{dy}{A_y} = \frac{dz}{A_z}.$$

Bu deňlemeler ulgamy $\boldsymbol{A}(P)$ meýdanyň wektorly çyzyklarynyň köplügi bolup, differensial deňlemeleriň ulgamydyr.

Eger meýdan tekiz bolsa, onda wektorly çyzyklar *Oxy* tekizlige parallel bolan tekizliklerde ýatýarlar we

$$\frac{dx}{A_x} = \frac{dy}{A_y}$$

deňleme tekizlikde wektorly çyzyklaryň deňlemesidir.

1-nji mysal. Goý, jisim hemişelik ω burçlaýyn tizlik bilen *Oz* okuň daşynda aýlanýan bolsun. Jisimiň çyzyklaýyn tizlikler meýdanynyň wektorly çyzygyny tapmaly.

Çözülişi. 1-nji mysaldan (§1) görnüşi ýaly jisimiň çyzyklaýyn tizlikler meýdany

$$\boldsymbol{A}(P) = -\omega x\boldsymbol{i} + \omega y\boldsymbol{j}$$

funksiýa bilen kesgitlenilýär. Bu meýdanyň wektorly çyzygynyň differensial deňlemesini ýazalyň:

$$\frac{dx}{-\omega y}=\frac{dy}{\omega x} \text{ ýa-da } \frac{dx}{-y}=\frac{dy}{x}.$$

Bu differensial deňlemäni çözüp, alýarys:

$$x^2+y^2=R, z=z_0=const.$$

Görnüşi ýaly, bu meýdanyň wektorly çyzyklary töwerekdir.

2-nji mysal. $\boldsymbol{z}=x\boldsymbol{i}+\boldsymbol{y}j+zj$ wektorly meýdanyň wektorly çyzygyny tapmaly.

Çözülişi. Wektorly çyzyklaryň differensial deňlemesini ýazalyň: $\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}.$

Bu differensial deňlemäni çözüp, alarys: $x=\frac{y}{C_1}=\frac{z}{C_2}.$

Görnüşi ýaly, bu wektorly meýdanyň wektorly çyzyklary koordinatalar başlangyjyndan geçýän gönülerdir.

3-nji mysal. Tükeniksiz uzynlykly geçirijidäki toguň magnit meýdanynyň wektorly çyzygynyň deňlemesini ýazmaly we ony çözmeli.

Çözülişi. Ýönekeýlik üçin geçiriji Oz ok bilen gabat gelýär we tok hem okuň ugry bilen ugrukdyrylan diýeliň. Tok bilen emele gelýän magnit meýdanynyň H wektor napryaženiýesi

$$\boldsymbol{H}=\frac{2}{\rho^2}[\boldsymbol{I}\times\boldsymbol{r}]$$

deňlik bilen tapylýar. Bu ýerde $\boldsymbol{I}=I\boldsymbol{k}$ wektor tok, $\boldsymbol{r}$ – $M(x,y,z)$ nokadyň radius-wektory, ρ – M nokat bilen geçirijiniň aralygyndaky burç. $[\boldsymbol{I}\times\boldsymbol{r}]$ – wektor köpeltmek hasylyny açyp, alýarys:

$$\boldsymbol{H}=\frac{2}{\rho^2}\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ 0 & 0 & I\\ x & y & z\end{vmatrix}=\frac{2}{\rho^2}(xI\boldsymbol{j}-\boldsymbol{i}yI)=-\frac{2Iy}{\rho^2}\boldsymbol{i}+\frac{2Ix}{\rho^2}\boldsymbol{j}.$$

Wektor çyzyklaryň differensial deňlemesini ýazalyň:

$$\frac{dx}{-\frac{2Iy}{\rho^2}}=\frac{dy}{\frac{2Ix}{\rho^2}}=\frac{dz}{0} \text{ ýa-da } \frac{dx}{-y}=\frac{dy}{x}=\frac{dz}{0}.$$

Bu ýerden $\begin{cases} x^2 + y^2 = R^2 \\ z = C \end{cases}$. Görnüşi ýaly wektor çyzyklar merkezi Oz okda bolan töwereklerdir.

39. $\boldsymbol{A}(P) = -y\boldsymbol{i} + x\boldsymbol{j} + b\boldsymbol{k}$ wektor meýdanyň (1,0,0) nokat arkaly geçýän wektorly çyzygynyň deňlemesini ýazmaly we ony çözmeli.

40. Eger a_1, a_2, a_3 hemişelik sanlar bolsa, $\boldsymbol{A}(P) = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ wektor meýdanyň, wektorly çyzygynyň deňlemesini ýazmaly we ony çözmeli.

41. $\boldsymbol{A}(P) = (z - y)\boldsymbol{i} + (x - z)\boldsymbol{j} + (y - x)\boldsymbol{k}$ wektor meýdanyň wektorly çyzygynyň deňlemesini ýazmaly.

Aşakdaky tekiz wektor meýdanlaryň wektor çyzyklaryny tapmaly.

42. $\boldsymbol{A}(P) = x\boldsymbol{i} + 2y\boldsymbol{j}$.
43. $\boldsymbol{A}(P) = x\boldsymbol{i} + z\boldsymbol{k}$.
44. $\boldsymbol{A}(P) = x\boldsymbol{i} - y\boldsymbol{j}$.
45. $\boldsymbol{A}(P) = 2z\boldsymbol{j} + 4y\boldsymbol{k}$.
46. $\boldsymbol{A}(P) = x^2\boldsymbol{i} + y^2\boldsymbol{j}$.
47. $\boldsymbol{A}(P) = z\boldsymbol{j} - y\boldsymbol{k}$.

§ 3. Wektoryň akymy

Goý,

$$\boldsymbol{A}(P) = A_x(x,y,z)\boldsymbol{i} + A_y(x,y,z)\boldsymbol{j} + A_z(x,y,z)\boldsymbol{k}.$$

wektorlary bolan wektorly meýdan berlen bolsun. Şu meýdanda berlen ikitaraply S üstüň bir tarapyna garalyň. Üstüň garalýan tarapynyň erkin nokadynda geçirilen normalyň birlik wektoryny $\boldsymbol{n}^0$ bilen belgiläliň. $\boldsymbol{n}^0$ wektoryň proýeksiýalary normalyň ugrukdyryjy kosinuslarydyr: $n^0\{\cos\alpha, \cos\beta, \cos\gamma\}$.Wektorly meýdanyň $\boldsymbol{A}(P)$ wektory bilen, $\boldsymbol{n}$ normalyň $\boldsymbol{n}^0$ birlik wektorynyň skalýar köpeltmek hasylyndan S üst boýunça alnan integrala garalyň:

$$\iint_S (\boldsymbol{A}(P), \boldsymbol{n}^0) dS = \iint_S (A_x \cos\alpha + A_y \cos\beta + A_z \cos\gamma) dS \qquad (1)$$

ýa-da

$$\iint_S (\boldsymbol{A}(P), \boldsymbol{n}^0) dS = \iint_S A_x(x,y,z)dydz + A_y(x,y,z)dxdz +$$
$$+ A_z(x,y,z)dxdy,$$

bu ýerde,

$$\cos\alpha dS = dydz, \cos\beta dS = dxdz, \cos\gamma dS - dxdy.$$

Eger $A(P)$ akýan suwuklyklaryň tizlikleriniň meýdany bolsa, (1) integral S üst boýunça suwuklyklaryň akymyny aňladýar. Erkin wektorly meýdanda (1) integrala S üst boýunça wektoryň akymy diýilýär we K bilen belgilenýär. Şunlukda, biz aşakdaky kesgitlemäni aldyk.

Kesgitleme. Wektorly meýdandaky S üste geçirilen normalyň birlik wektory bilen berlen $A(P)$ wektoryň skalýar köpeltmek hasylyndan S üst boýunça alnan birinji görnüşli üst integrala wektoryň üst boýunça akymy diýilýär we aşakdaky ýaly belgilenýär:

$$K = \iint_S \boldsymbol{A}(P)\boldsymbol{n} dS = \iint_S (\boldsymbol{A}(P), \boldsymbol{n}^0) dS.$$

Şunlukda, wektoryň üst boýunça akymyny hasaplamak üçin üst boýunça alnan birinji görnüşli integraly hasaplamaly bolýarys. Diýmek, K wektor akym skalýar ululykdyr. Belli bolşy ýaly, $\boldsymbol{A}(P)$ wektoryň n normal wektorynyň birlik wektoryna skalýar köpeltmek hasylynyň $\boldsymbol{A}(P)$ wektoryň n ugur boýunça $A_n(P)$ proýeksiýasyna deňligine görä, K akymy

$$K = \iint_S A_n(P) d\sigma$$

görnüşinde ýazyp bileris. Eger üstüň käbir böleginde $\boldsymbol{A}(P)$ wektor hemişelik bolsa, $\boldsymbol{A}(P) = A_n = const$, onda üstüň Q bolan bölegindäki akýan akymy A_nQ bolar.

Eger $A(P)$ wektorly meýdan suwuklyklaryň tizlikleriniň meýdany bolsa, onda K akym şu meýdana girýän suwuklyk bilen çykýan suwuklygyň tapawudyna deňdir.

Eger K=0 bolsa, onda berlen ýaýla näçe suwuklyk gelýän bolsa, şonça suwuklyk hem çykýandyr.

Eger K>0 bolsa, onda bu ýaýla gelýän suwuklyk çykýan suwuklykdan kändir. Ýagny bu oblastda çeşme bardyr. K<0 bolanda bolsa tersine.

Wektor akymyň birnäçe ýönekeý häsiýetlerine garap geçeliň.

1. Eger üste geçirilen n normalyň ugruny üýtgetsek, ýagny üstüň aşaky üstüne geçsek wektor akym alamatyny üýtgedýär:

$$\iint_{S^+} (A(P),n)dS = -\iint_{S^-} (A(P),n)dS.$$

2. Akymyň çyzyklylyk häsiýeti.

$$\iint_S (\lambda A(P),\boldsymbol{n} + \mu B(P),\boldsymbol{n})dS = \lambda \iint_S (A(P),\boldsymbol{n})dS + \mu \iint_S (B(P)),\boldsymbol{n})dS$$

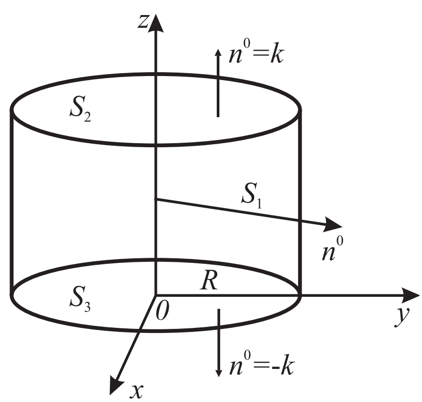
bu ýerde λ we μ hemişelik sanlardyr.

3. Eger S üst $S_1, S_2, ..., S_m$ endigan üstleriň jeminden ybarat bolsa, onda $A(P)$ wektoryň S üst boýunça akymy, $S_1, S_2, ..., S_m$ üstler boýunça akymlarynyň jemine deňdir:

$$K = \sum_{i=1}^{m} \iint_{S_i} (A(P),n)dS.$$

1-nji mysal. $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ radius-wektoryň $x^2 + y^2 \le R^2$, $0 \le z \le H$ göni silindriň doly üsti boýunça akymyny tapmaly.

Çözülişi. Silindriň doly S üsti, onuň S_1 gapdal üstünden, S_2 ýokarky üstünden we S_3 aşaky üstünden ybaratdyr. Onda, K_1, K_2, K_3 degişlilikde S_1, S_2, S_3 üstlerden akyp geçýän akymlar bolsa, K_1, K_2, K_3 $K = K_1 + K_2 + K_3$ deňlik dogrudyr.



6-njy surat

Silindriň S_1 gapdal üstüne geçirilen normalyň xOy tekizlige perpendikulýardygyna görä

$$(\boldsymbol{A}(P),\boldsymbol{n}) = (\boldsymbol{r},\boldsymbol{n}) = pr_n r = R.$$

Onda

$$K_1 = \iint_{S_1} (\boldsymbol{r}, \boldsymbol{n}) dS = R \iint_{S_1} dS = 2\pi\sqrt{R^2 H}.$$

Silindriň S_2 ýokarky üstüne geçirilen n normalyň Oz oka paralleldigine görä,

$$(\boldsymbol{A}(P), \boldsymbol{n}) = (\boldsymbol{r}, \boldsymbol{n}) = pr_{0z} r = H.$$

Onda

$$K_2 = \iint_{S_2} (\boldsymbol{r}, \boldsymbol{n}) dS = H \iint_{S_2} dS = \pi R^2 H.$$

Silindriň S_3 aşaky üstünde $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ wektoryň $\boldsymbol{n} = -\boldsymbol{k}$ normala perpendikulýardygyna görä $(\boldsymbol{r}, -\boldsymbol{n}) = (\boldsymbol{r}, -\boldsymbol{k}) = 0$. Onda

$$K_3 = \iint_{S_3} (\boldsymbol{A}(P), \boldsymbol{n}) dS = 0.$$

Şunlukda, $K = K_1 + K_2 + K_3 = 2\pi R^2 H + \pi R^2 H + 0 = 3\pi R^2 H$.

2-nji mysal. Absissalar okuna perpendikulýar bolan, taraplary 2 we 3 bolan gönüburçluk formaly meýdandan $\boldsymbol{A}(P) = 2\boldsymbol{i}$ wektoryň Ox okuň oňyn ugry boýunça akymyny tapmaly.

Çözülişi. Wektoryň üst boýunça akymynyň kesgitlemesiniň esasynda, alarys:

$$K = \iint_S (\boldsymbol{A}(P), \boldsymbol{n}) dS.$$

Bu ýerde $\boldsymbol{A}(P) = 2\boldsymbol{i}$, $n = \boldsymbol{i}$, $(\boldsymbol{A}(P, n)) = (2\boldsymbol{i}, \boldsymbol{i}) = 2$. Gönüburçlugyň meýdanynyň 6 deňligine görä, alarys:

$$K = \iint_S 2dS = 2 \cdot 6 = 12.$$

3-nji mysal. $\boldsymbol{A}(P) = \dfrac{r_1}{|r|^3}$ wektorly meýdanyň, merkezi koordinatalar başlangyjynda, radiusy R bolan sferadan akyp geçýän akymyny tapmaly.

Çözülişi. Sfera geçirilen $\boldsymbol{n}$ normalyň $\boldsymbol{r}_1$ radius-wektora kollineardygyna görä normalyň $\boldsymbol{n}^0$ birlik wektoryny $\boldsymbol{n}^0 = \frac{r_1}{|r|}$ görnüşde almak bolar. Şonuň üçin

$$(A(P), \boldsymbol{n}^0) = \left(\frac{r_1}{|r|^3}, \frac{r_1}{|r|}\right) = \frac{1}{|r|^4}(\boldsymbol{r}_1, \boldsymbol{r}_1) = \frac{|r_1|^2}{|r|^4} = \frac{1}{|r|^2}.$$

S sferada $|r| = R$ bolýandygyna görä $(\boldsymbol{A}(P), \boldsymbol{n}^0) = \frac{1}{R^2}$.

Onda

$$K = \iint_S (\boldsymbol{A}(P), \boldsymbol{n}^0) dS = \frac{1}{R^2} \iint_S dS = 4\pi, \ \left(\oiint_S dS = 4\pi R^2\right).$$

4-nji mysal. $\boldsymbol{A}(P) = x^2\boldsymbol{i} + y^2\boldsymbol{j} + z^2\boldsymbol{k}$ wektorly meýdanyň $(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2$ sferanyň daşky S_+ tarapyndan akyp geçýän akymyny tapmaly.

Çözülişi. Wektorly meýdanyň akymynyň kesgitlemesiniň esasynda,

$$K = \iint_{S_+} (x^2 \cos\alpha + y^2 \cos\beta + z^2 \cos\gamma) dS.$$

ikinji görnüşli üst integralyny hasaplamaly bolýarys. $\boldsymbol{N}$ normaly we sferanyň daşky tarapydygyny aňladýan normalyň $\boldsymbol{n}$ birlik wektoryny ýazalyň:

$$\boldsymbol{N} = \{2(x-a), 2(y-b), 2(z-c)\},$$

$$\boldsymbol{n} = \left\{\frac{x-a}{R}, \frac{y-b}{R}, \frac{z-c}{R}\right\}.$$

Onda berlen integraly aşakdaky görnüşde ýazyp bileris:

$$K = \iint_{S_+} \left(x^2 \frac{x-a}{R} + y^2 \frac{y-b}{R} + z^2 \frac{z-c}{R}\right) dS.$$

Integraly çözmek üçin sferanyň parametrik deňlemesini ýazalyň:

$$x = a + R\cos v\sin u,\ y = b + R\sin v\sin u,\ z = c + R\cos u,$$

$$0 \le u \le \pi, 0 \le v \le 2\pi, \sqrt{EG - F^2} = R^2.$$

Onda

$$K = \int_0^{2\pi} dv \int_0^{\pi} R^2 \sin u[(a + R\cos v\sin u)^2 \cos v\sin u +$$

$$+(b + R\sin v\sin u)^2 \sin v\sin u + (c + R\cos u)^2 \cos u]du =$$

$$= R^2 \int_0^{2\pi} 2aR\cos^2 v dv \int_0^{\pi} \sin^3 u du + R^2 \int_0^{2\pi} 2br\sin^2 v \int_0^{\pi} \sin^3 u du +$$

$$+R^2 \int_0^{2\pi} 2cRdv \int_0^{\pi} \cos^2 u\sin^2 u du = \frac{8\pi R^3}{3}(a + b + c).$$

48. Koordinatalar başlangyjynda ýerleşdirilen položitel e zarýad güýjenme wektorly meýdany döredýär. Ýagny giňişligiň her bir nokadyndaky $\boldsymbol{F}$ wektor Kulonyň kanuny boýunça

$$\boldsymbol{F} = k\frac{e}{r^2}\boldsymbol{r}_1$$

deňlik bilen tapylýar. Bu ýerde $\boldsymbol{r}$ berlen nokat bilen koordinatalar başlagyjynyň aralygyndaky uzaklyk, $\boldsymbol{r}_1$ radius-wektoryň ugry boýunça ugrukdyrylan birlik wektor, k bolsa hemişelik koeffisiýentdir. Merkezi koordinatalar başlangyjynda, radiusy R bolan sferadan akyp geçýän wektorly meýdanyň akymyny tapmaly.

49. Depeleri $M(1,2,0)$, $N(0,2,0)$, $L(0,2,2)$ nokatlarda bolan üçburçluk görnüşli meýdandan, $\boldsymbol{A}(P)=3\boldsymbol{j}$ wektoryň koordinatalar başlangyjyna görä ugur boýunça akymyny tapmaly.

50. $\boldsymbol{A}(P) = x^3\boldsymbol{i} + y^3\boldsymbol{j} + z^3\boldsymbol{k}$ wektorly meýdanyň $x^2 + y^2 + z^2 = x$ sferanyň daşky S_+ tarapyndan akyp geçýän akymyny tapmaly.

51. $\boldsymbol{A}(P) = x^2\boldsymbol{i} + y^2\boldsymbol{j} + z^2\boldsymbol{k}$ wektorly meýdanyň $x^2+y^2+z^2=1$, $x \ge 0, y \ge 0, z \ge 0$ sferanyň ýokarky bölegi boýunça akymyny tapmaly.

52. $\boldsymbol{A}(P)x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ wektorly meýdanyň $z = 1 - \sqrt{x^2 + y^2}$, $0 \leq z \leq 1$, üst boýunça akymyny tapmaly.

53. Goý, esasynyň radiusy R bolan silindr görnüşli jisim Oz oka perpendikulýar bolsun. Eger α, β, γ hemişelik sanlar bolsa, jisimiň meýdany arkaly Oz okuň ugry boýunça $\boldsymbol{A}(P) = \alpha\boldsymbol{i} + \beta\boldsymbol{j} + \gamma\boldsymbol{k}$ wektoryň akymyny tapmaly.

54. Depesi koordinatalar başlangyjynda, esasynyň radiusy R we beýikligi h bolan konusyň daşky tarapy boýunça wektoryň akymyny tapmaly.

§ 4. Wektoryň akymyny hasaplamagyň usullary

1. Koordinatalar tekizlikleriniň birisine proýektirlemek usuly.

Goý, S üst xOy tekizligiň D_{xy} ýaýlasyna proýektirlenen bolsun. Onda S üstüň deňlemesi $z = f(x, y)$ görnüşdedir. Üstüň dS elementiniň meýdanynyň

$$dS = \frac{dxdy}{|\cos\gamma|}$$

deňdigine görä, S üstüň saýlanyp alnan tarapyndaky akymyny hasaplamak üçin ikigat integraly hasaplamaly bolýarys:

$$K = \iint_S (\boldsymbol{A}(P), \boldsymbol{n})dS = \iint_{D_{xy}} \left.\frac{(\boldsymbol{A}(P), \boldsymbol{n})}{\cos\gamma}\right|_{z=f(x,y)} dxdy. \tag{1}$$

$\boldsymbol{n}$ normalyň $\boldsymbol{n}^0$ birlik wektory aşakdaky formula bilen tapylýar:

$$\boldsymbol{n}^0 = \pm\frac{\operatorname{grad}[z - f(x,y)]}{|\operatorname{grad}[z - f(x,y)]|} = \pm\frac{-\dfrac{\partial f}{\partial x}\boldsymbol{i} - \dfrac{\partial f}{\partial y}\boldsymbol{j} + \boldsymbol{k}}{\sqrt{\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2 + 1}}. \tag{2}$$

$\cos\gamma$ bolsa (2) formuladaky $\boldsymbol{k}$ ortyň öňündäki koeffisiýente deňdir:

$$\cos\gamma = \pm\frac{1}{\sqrt{\left(\dfrac{\partial f}{\partial x}\right)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2 + 1}} \tag{3}$$

Eger Oz ok bilen $\boldsymbol{n}$ normalyň aralygyndaky burç ýiti burç bolsa, (2) we (3) formulalarda “+” alamaty, tersine bolanda bolsa “–” alamaty alynýar.

Şeýle hem S üsti yOz ýa-da xOz tekizliklere proýektirläp (1) formula meňzeş bolan formulalary alyp bolýar.

5-nji mysal. $\boldsymbol{A}(P) = y\boldsymbol{i} + z\boldsymbol{j} + x\boldsymbol{k}$ wektorly meýdanyň $x+y+z=a$, x=0, 1=0, y=0, tekizlikler bilen çäklenen piramidanyň ýokarky tarapyndan akyp geçýän akymyny tapmaly.

Çözülişi. $x+y+z=a$ piramidanyň tekizlikleri berlen üçburçluklaryňyň ýerleşen tekizlikleridir.

$z=a–x–y$. Bu piramidanyň xOy tekizlige D_{xy} proýeksiýasy hem üçburçlukdyr.

$$\boldsymbol{n}^0 = \frac{\operatorname{grad}(x+y+z-a)}{|\operatorname{grad}(x+y+z-a)|} = \frac{1}{\sqrt{3}}\boldsymbol{i} + \frac{1}{\sqrt{3}}\boldsymbol{j} + \frac{1}{\sqrt{3}}\boldsymbol{k}.$$

Skalýar köpeltmek hasylyny tapalyň:

$$(\boldsymbol{A}(P), \boldsymbol{n}^0) = \left(y\boldsymbol{i} + z\boldsymbol{j} + x\boldsymbol{k}, \frac{1}{\sqrt{3}}(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}\right) = \frac{x+y+z}{\sqrt{3}}.$$

Şeýle hem

$$\cos\gamma = \frac{1}{\sqrt{3}}, dS = \frac{dxdy}{|\cos\gamma|} = \sqrt{3dxdy}.$$

Wektoryň akymyny hasaplamak üçin (1) formulany ulanalyň:

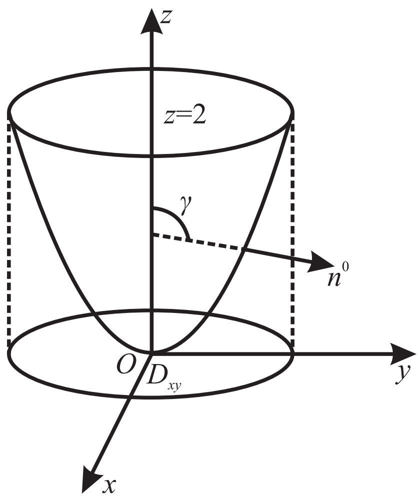
$$K = \iint\limits_S (\boldsymbol{A}(P), \boldsymbol{n}^0)dS = \iint\limits_{D_{xy}} (x+y+z)\Big|_{z=a-x-y} dxdy = a\iint\limits_{D_{xy}} dxdy =$$

$$= a\int\limits_0^a dx \int\limits_0^{a-x} dy = \frac{a^3}{3}.$$

6-njy mysal. $\boldsymbol{A}(P) = y^2\boldsymbol{j} + \boldsymbol{k}z$ wektor meýdanyň $z = x^2 + y^2$ paraboloidanyň $z = 2$ tekizlik bilen bölünip alnan böleginden akyp geçýän akymyny tapmaly.

Çözülişi. Berlen ýaýlanyň (aýlanma paraboloid) xOy tekizlige bolan proýeksiýasyny D_{xy} bilen belgiläliň. S üste geçirilen normalyň $\boldsymbol{n}^0$ birlik wektoryny tapalyň:

$$\boldsymbol{n}^0 = \pm\frac{\operatorname{grad}[z - x^2 - y^2]}{|\operatorname{grad}(z - x^2 - y^2)|} = \pm\frac{-2x\boldsymbol{i} - 2y\boldsymbol{j} + \boldsymbol{k}}{\sqrt{4x^2 + 4y^2 + 1}}.$$



7-nji surat

Meseläniň şertine görä $\boldsymbol{n}^0$ birlik wektor Oz ok bilen kütek burçy

emele getirýär. Şonuň üçin drobuň öňünde “–” alamaty alynýar.

$$\boldsymbol{n}^0 = \frac{2x\boldsymbol{i} + 2y\boldsymbol{j} - \boldsymbol{k}}{\sqrt{4x^2 + 4y^2 + 1}}, \ \cos\gamma = -\frac{1}{\sqrt{4x^2 + 4y^2 + 1}} \prec 0,$$

Onda

$$dS = \frac{dxdy}{|\cos\gamma|} = \sqrt{4x^2 + 4y^2 + 1dxdy}.$$

Skalýar köpeltmek hasylyny tapalyň:

$$(\boldsymbol{A}(P), \boldsymbol{n}^0) = \frac{2y^3 - z}{\sqrt{4x^2 + 4y^2 + 1}}.$$

Şunlukda,

$$K = \iint\limits_{D_{xy}} (\boldsymbol{A}(P), \boldsymbol{n}^0) dS = \iint\limits_{D_{xy}} (2y^3 - z)\big|_{z = x^2 + y^2} dxdy =$$

$$= \iint\limits_{D_{xy}} (2y^3 - y^2 - x^2) dxdy.$$

D_{xy} integrirleme ýaýlasy merkezi koordinatalar başlangyjynda, radiusy $R = \sqrt{2}$ bolan töwerekdir. Polýar koordinatalaryna geçýäris: $x = \rho\cos\varphi, y = \rho\sin\varphi$.

$$K = \iint\limits_{D_{xy}} (2\rho^3\sin^3\varphi - \rho^2)\rho d\rho d\varphi = \int\limits_0^{2\pi} d\varphi \int\limits_0^{\sqrt{2}} (2\rho^4\sin^3\varphi - \rho^3)d\rho =$$

$$= -2\pi\frac{\rho^4}{4}\Big|_0^{\sqrt{2}} = -2\pi.$$

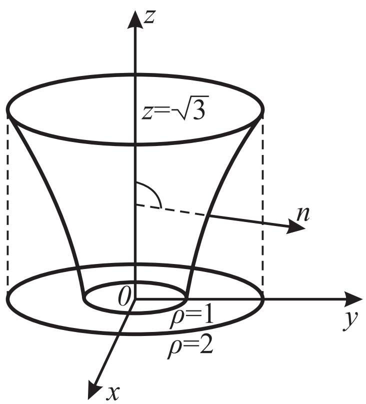
7-nji mysal. $\boldsymbol{A}(P) = x\boldsymbol{i} + y\boldsymbol{j} + \sqrt{x^2 + y^2 - 1}\,\boldsymbol{k}$ wektorly meýdanyň $z = 0,\ z = \sqrt{3},\ z = \sqrt{x^2 + y^2 - 1}$ giperboloidanyň daşky tarapy boýunça akymyny tapmaly.

Çözülişi. Berlen üst xOy tekizligiň D_{xy} ýaýlasyna proýektirlenen. D_{xy} ýaýla bolsa

$$\begin{cases} x^2 + y^2 = 1, \\ z = 0; \end{cases} \quad \text{we} \quad \begin{cases} x^2 + y^2 = 1, \\ z = 4 \end{cases}$$

töwerekler bilen çäklenendir. Daşky $\boldsymbol{n}$ normaly tapalyň:

$$\boldsymbol{n} = \pm\, grad(z - \sqrt{x^2 + y^2 - 1}) = \pm\left(\frac{-x\boldsymbol{i} - y\boldsymbol{j}}{\sqrt{x^2 + y^2 - 1}} + \boldsymbol{k}\right)$$



8-nji surat

$\boldsymbol{n}$ normal Oz ok bilen kütek burçy emele getirýär. Şonuň üçin "–" alamatyny alýarys:

$$\boldsymbol{n} = \frac{x\boldsymbol{i} + y\boldsymbol{j}}{\sqrt{x^2 + y^2 - 1}} - \boldsymbol{k}.$$

Skalýar köpeltmek hasylyny tapalyň:

$$(\boldsymbol{A}(P), \boldsymbol{n}) = \frac{x^2 + y^2}{\sqrt{x^2 + y^2 - 1}} - \sqrt{x^2 + y^2 - 1} = \frac{1}{\sqrt{x^2 + y^2 - 1}}.$$

Onda

$$K = \iint_S (\boldsymbol{A}(P), \boldsymbol{n}) dS = \iint_{D_{xy}} \frac{dxdy}{\sqrt{x^2 + y^2 - 1}}.$$

$x = \rho \cos\varphi, y = \rho \sin\varphi$ polýar koordinatalaryna geçip, alýarys:

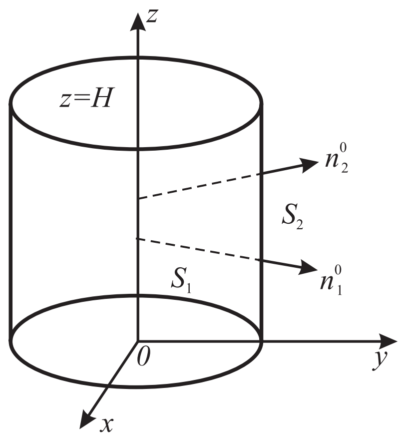
$$K = \iint_{D_{xy}} \frac{\rho d\varphi d\rho}{\sqrt{\rho^2 - 1}} = \int_0^{2\pi} d\varphi \int_1^2 \frac{\rho d\rho}{\sqrt{\rho^2 - 1}} = 2\pi \sqrt{\rho^2 - 1}\Big|_1^2 = 2\sqrt{3}\pi.$$

8-nji mysal. $\boldsymbol{A}(P) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ wektorly meýdanyň z=0, z=H tekizlikler bilen çäklenen $x^2 + y^2 = R^2$ tegelek silindriň daşky tarapyndan akyp geçýän akymyny tapmaly.

Çözülişi. Berlen silindriň xOy tekizlige bolan proýeksiýasy

$$\begin{cases} x^2 + y^2 = R^2, \\ z = 0 \end{cases}$$

töwerekdir.



9-njy surat

Şonuň üçin silindri beýleki koordinatalar tekizliklerine, mysal üçin yOz tekizlige proýektirleýäris. Onda onuň daşky tarapyndan akyp geçýän akym $K = K_1 + K_2$ bolar. Bu ýerde K_1 akym $S_{1,y} \geq 0$ üstden akyp geçýän akym, K_2 bolsa $S_{2,y} < 0$ üstden akyp geçýän akymdyr.

S_1 üste geçirilen normalyň birlik wektoryny we skalýar köpeltmek hasylyny tapalyň:

$$\boldsymbol{n}^0 = \frac{x\boldsymbol{i} + y\boldsymbol{j}}{R}, (\boldsymbol{A}(P), \boldsymbol{n}^0) = \frac{x^2 + y^2}{R} = R,$$

Onda

$$K_1 = \iint_{S_1} Rds = R\iint_{S_1} ds = R\pi RH = \pi R^2 H.$$

Şeýle hem

$$\boldsymbol{n}^0 = \frac{x\boldsymbol{i} + y\boldsymbol{j}}{R}, \quad (\boldsymbol{A}(P), \boldsymbol{n}^0) = \frac{x^2 + y^2}{R} = R,$$

$$K_2 = \iint_{S_2} Rds = RS_2 = \pi R^2 H.$$

Şunlukda, $K = K_1 + K_2 = \pi R^2 H + \pi R^2 H = 2\pi R^2 H.$

55. Depeleri $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$ nokatlarda bolan ABC üçburçlugyň ýokarky tarapy boýunça $\boldsymbol{A}(P) = (x - 2z)\boldsymbol{i} + (x + 3y + z)\boldsymbol{j} + (5x + y)\boldsymbol{k}$ wektorly meýdanyň akymyny tapmaly.

56. $\boldsymbol{A}(P) = xz\boldsymbol{i}$ wektorly meýdanyň $z = 1 - x^2 - y^2$ paraboloidanyň z=0 tekizlik bilen çäklenen daşky tarapy boýunça akymyny tapmaly.

57. $\boldsymbol{A}(P) = \frac{x}{2}\boldsymbol{i} + \frac{y}{2}\boldsymbol{j}$ wektorly meýdanyň $z = x^2 + y^2, 0 \leq z \leq 1$ paraboloidanyň ýokarky tarapy boýunça akymyny tapmaly.

58. $\boldsymbol{A}(P)=yz\boldsymbol{i}+xz\boldsymbol{j}+xy\boldsymbol{k}$ wektorly meýdanyň $z=0$, $z=H$ tekizlikler bilen çäklenen $x^2+y^2=R^2$ silindriň daşky tarapy boýunça akymyny tapmaly.

59. $\boldsymbol{A}(P)=x^3\boldsymbol{i}+y^3\boldsymbol{j}+z^3\boldsymbol{k}$ wektorly meýdanyň $\dfrac{x^2+y^2}{R^2}\le\dfrac{z^2}{H^2}$, $0\le z\le H$ konusyň üstu boýunça akymyny tapmaly.

60. $\boldsymbol{A}(P)=yz\boldsymbol{i}-x\boldsymbol{j}-y\boldsymbol{k}$ wektorly meýdanyň $z=1(0\le z\le 1)$ tekizlik bilen çäklenen, $x^2+y^2=z^2$ konusyň doly üsti boýunça akymyny tapmaly.

2. Koordinatalar tekizliklerine proýektirlemek usuly.

Goý, berlen S üst koordinatalar tekizlikleriň üçüsine-de proýektirlenýän bolsun. Onuň xOy, xOz, yOz tekizliklere bolan proýeksiýalaryny degişlilikde D_{xy}, D_{xz}, D_{yz} bilen belgiläliň.

Bu halda S üstüň $F(x,y,z)=0$ deňlemesi x,y,z ululyklaryň her birisine görä çözülýändir:

$$x=x(y,z),\ \ y=y(x,z),\ \ z=z(x,y).$$

Onda normalyň $\boldsymbol{n}^0$ birlik wektory

$$\boldsymbol{n}^0=\boldsymbol{i}\cos\alpha+\boldsymbol{j}\cos\beta+\boldsymbol{k}\cos\gamma.$$

bolan, S üstden akyp geçýän

$$\boldsymbol{A}(P)=P_1(x,y,z)\boldsymbol{i}+Q(x,y,z)\boldsymbol{j}+R(x,y,z)\boldsymbol{k}.$$

wektorly meýdanyň akymy aşakdaky deňlik bilen tapylýar:

$$K=\pm\iint(\boldsymbol{A}(P),\boldsymbol{n}^0)dS=\iint\limits_S[P(x,y,z)\cos\alpha+Q(x,y,z)\cos\beta+ \qquad (1)$$
$$+R(x,y,z)\cos\gamma]dS.$$

Bu ýerde

$$dS\cos\alpha=\pm\,dydz, dS\cos\beta=\pm\,dxdz, dS\cos\gamma=\pm\,dxdy \qquad (2)$$

bolýandygyna görä (1) formulanyň her birisiniň alamaty $\cos\alpha, \cos\beta, \cos\gamma$ burçlaryň alamaty bilen alynýar. Onda

$$K = \pm \iint\limits_{D_{yz}} P[x(y,z), yz]dydz \pm \iint\limits_{D_{xz}} Q[x, y(x,z), z]dxdz \pm$$
$$\pm \iint\limits_{D_{xy}} R[x, y, z(x,y)]dxdy.$$

9-njy mysal. $\boldsymbol{A}(P) = x\boldsymbol{i} + \boldsymbol{j} + xz^2\boldsymbol{k}$ wektorly meýdanyň $x^2 + y^2 + z^2 = 1$ sferanyň birinji oktantynda ýerleşen bölegininiň daşky tarapyndan akyp geçýän akymyny tapmaly.

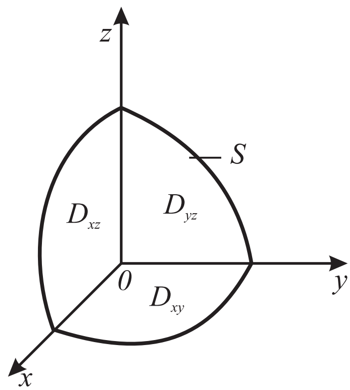
Çözülişi. Alýarys:

$$\boldsymbol{n}^0 = \frac{\mathrm{grad}(x^2 + y^2 + z^2 - 1)}{|\,\mathrm{grad}(x^2 + y^2 + z^2 - 1)\,|} = \frac{x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}}{\sqrt{x^2 + y^2 + z^2}} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}.$$

Üstüň birinji oktanda-da ýerleşýändigi üçin $\cos\alpha x \geq 0$, $\cos\beta = y \geq 0, \cos\gamma = z \geq 0$ we meselänuň şertine görä $P_1 = x, Q = 1, R = xz^2$.

Onda

$$K = \iint\limits_{D_{yz}} xdydz + \iint\limits_{D_{xz}} dxdy + \iint\limits_{D_{xdy}} xz^2 dxdy = K_1 + K_2 + K_3.$$



10-njy surat

$$K_1 = \iint\limits_{D_{yz}} \sqrt{1 - x^2 - y^2}\, dydz, K_2 = \iint\limits_{D_{xz}} dxdz, K_3 = \iint xz^2 dxdy =$$
$$= \iint\limits_{D_{xy}} x(1 - 2^2 - y^2)dxdy.$$

Ikinji integral D_{yz} ýaýlanyň meýdanyna deňdir. Birinji we üçünji integrallary polýar koordinatalara geçip hasaplaýarys:

$$K_1=\int_0^{\frac{\pi}{2}}d\varphi\int_0^1\sqrt{1-\rho^2}\rho d\rho=\frac{\pi}{2}\left(-\frac{(1-\rho^2)^{\frac{3}{2}}}{3}\right)\Bigg|_0^1=\frac{\pi}{6}.$$

$$K_3=\int_0^{\frac{\pi}{2}}\cos\varphi d\varphi\int_0^1\rho(1-\rho^2)\rho d\rho=1\left(\frac{r^3}{3}-\frac{r^5}{5}\right)\Bigg|_0^1=\frac{2}{15}.$$

Şunlukda,

$$K=K_1+K_2+K_3=\frac{\pi}{4}+\frac{\pi}{6}+\frac{2}{15}=\frac{5\pi}{12}+\frac{2}{15}.$$

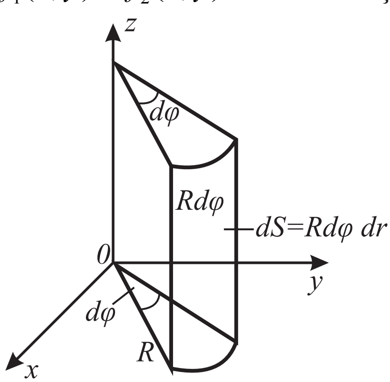
61. $\boldsymbol{A}(P)=xy\boldsymbol{i}+yz\boldsymbol{j}+xz\boldsymbol{k}$ wektorly meýdanyň $x^2+y^2+z^2=1$ sferanyň birinji oktantynda ýerleşen böleginiň daşky tarapyndan akyp geçýän akymyny tapmaly.

62. Koordinatalar tekizliklerine proýektirlemek usulyny ulanyp, $\boldsymbol{A}(P)=z\boldsymbol{i}-x\boldsymbol{j}+y\boldsymbol{k}$ wektorly meýdanyň, deňlemesi $3x+6y-2z-6=0$ bolan tekizligiň koordinatalar tekizlikleri bilen kesişmeginden alnan, üçburçlugyň ýokarky tarapy boýunça akyp geçýän akymyny tapmaly.

3. Egriçyzykly koordinatalar ulgamyny girizmek usuly.

Seredip geçen bölümçelerimizde biz berlen S üsti koordinata tekizligine ýa-da koordinatalar tekizliklerine proýektirlemek bilen üstden akyp geçýän akymy hasaplapdyk. Kähalatlarda üstde ýönekeý koordinatalar ulgamyny girizmek bilen üstden akyp geçýän akymy aňsat hasaplap bolýar.

1. Goý, S üst $x^2+y^2=R^2$ tegelek silindriň $z=f_1(x,y)$ we $z=f_2(x,y)$, $f_1(x,y)\le f_2(x,y)$ üstler bilen çäklenen bölegi bolsun.



11-nji surat

$x = R\cos\varphi$, $y = R\sin\varphi$, $z = z$ silindr koordinatalar ulgamyny girizmek bilen berlen üst üçin ýazyp bileris:

$$0 \le \varphi \le 2\pi;$$

$$f_1(R\cos\varphi, R\sin\varphi) \le z \le f_2(R\cos\varphi, R\sin\varphi).$$

Üstüň dS elementi üçin aşakdaky aňlatmany alýarys:

$$dS = Rd\varphi\ dz.$$

Onda S üstüň daşky tarapyndan akyp geçýän wektorly meýdanyň akymy aşakdaky formula bilen hasaplanylýar:

$$K = R\int_0^{2\pi} d\varphi \int_{f_1(R\cos\varphi, R\sin\varphi)}^{f_2(R\cos\varphi, R\sin\varphi)} (\boldsymbol{A}(P, \boldsymbol{n}^0)dz.$$

Bu ýerde

$$\boldsymbol{n}^0 = \frac{\mathrm{grad}(x^2 + y^2 - R^2)}{\left|\mathrm{grad}(x^2 + y^2 - R^2)\right|} = \frac{x\boldsymbol{i} + y\boldsymbol{j}}{R}.$$

10-njy mysal. $\boldsymbol{A}(P) = y\boldsymbol{i} + x\boldsymbol{j} - e^{xyz}\boldsymbol{k}$ wektorly meýdanyň $x^2 + y^2 = R^2$ silindriň z=0 we x+y+z=4 tekizlikler bilen çäklenen böleginiň gapdal üstüniň daşky tarapy boýunça akyp geçýän akymyny tapmaly.

Çözülişi. Bu halda $R = 2$, $f_1(x, y) = 0$, $f_2(x, y) = 4 - x - y$.

Silindr koordinatalary girizeliň: $x = R\cos\varphi$, $y = R\sin\varphi$, $z = z$.

Onda $f_1(x, y) = 0$, $f_2(x, y) = 4 - 2\cos\varphi - 2\sin\varphi.$

$$K = \int_0^{2\pi} d\varphi \int_0^{4-2\cos\varphi-2\sin\varphi} (\boldsymbol{A}(P), \boldsymbol{n}^0)dz.$$

$$\boldsymbol{n}^0 = \frac{x\boldsymbol{i} + y\boldsymbol{j}}{2} = \boldsymbol{i}\cos\varphi + \boldsymbol{j}\sin\varphi,$$

$$(\boldsymbol{A}(P), \boldsymbol{n}^0) = 2\cos\varphi\sin\varphi + 2\cos\varphi\sin\varphi = 4\cos\varphi\sin\varphi,$$

$$K=\int_0^{2\pi}d\varphi\int_0^{4-2\cos\varphi-2\sin\varphi}4\cos\varphi\sin\varphi d\varphi=$$

$$=\int_0^{2\pi}4\cos\varphi\sin\varphi(4-2\cos\varphi-2\sin\varphi)d\varphi=0.$$

11-nji mysal. $\boldsymbol{A}(P)=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$ radius wektoryň $x^2+y^2=1$ silindriň, aşakdan $x+y+z=1$ tekizlik bilen, ýokardan $x+y+z=2$ tekizlik bilen çäklenen bölegininň gapdal üstündäki akymyny tapmaly.

Çözülişi. Bu mysal üçin

$R=1$, $f_1(x,y)=1-x-y$, $f_2(x,y)=2-x-y$.

Silindr koordinatalaryna geçeliň: $x=\cos\varphi$, $y=\sin\varphi$, $z=z$.

Onda

$f_1(x,y)=1-\cos\varphi-\sin\varphi$, $f_2(x,y)=2-\cos\varphi-\sin\varphi$,

$$K=\int_0^{2\pi}d\varphi\int_{1-\cos\varphi-\sin\varphi}^{2-\cos\varphi-\sin\varphi}(\boldsymbol{A}(P),\boldsymbol{n}^0)dz.$$

$x^2+y^2=1$ deňligi nazara almak bilen ýazyp bileris:

$\boldsymbol{n}^0=x\boldsymbol{i}+y\boldsymbol{j}=\boldsymbol{i}\cos\varphi+\boldsymbol{j}\sin\varphi$

$(\boldsymbol{A}(P),\boldsymbol{n}^0)=x^2+y^2=\cos^2\varphi+\sin^2\varphi=1$.

Şunlukda,

$$K=\int_0^{2\pi}d\varphi\int_{1-\cos\varphi-\sin\varphi}^{2-\cos\varphi-\sin\varphi}dz=\int_0^{2\pi}d\varphi=2\pi.$$

63. $\boldsymbol{A}(P)=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$ wektorly meýdanyň $x^2+y^2=R^2$ tegelek silindriň z=0 we z=H tekizlikler bilen çäklenen bölegininň gapdal üstüniň daşky tarapy boýunça akymyny tapmaly.

64. $\boldsymbol{A}(P) = x\boldsymbol{i} - y\boldsymbol{j} - xyz^3\boldsymbol{k}$ wektorly meýdanyň $x^2 + y^2 = 1$ silindriň z=0 we $z = x^2 - y^2$ giperbola paraboloidi bilen çäklenen bölegininň gapdal üstüniň daşky tarapy boýunça akymyny tapmaly.

65. $\boldsymbol{A}(P) = x\boldsymbol{i} - xy\boldsymbol{j} + z\boldsymbol{k}$ wektorly meýdanyň $x^2 + y^2 = R^2$ silindriň y=1 we $x + y = 4$ tekizlikler bilen çäklenen böleginiň daşky tarapyndaky akymyny tapmaly.

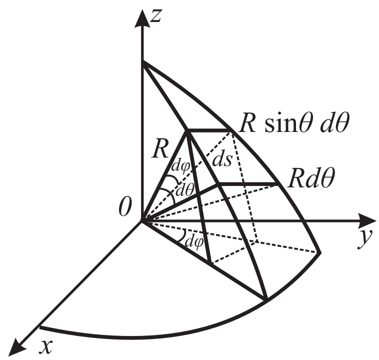
66. $\boldsymbol{A}(P) = x^3\boldsymbol{i} - y^3\boldsymbol{j} + xz^3\boldsymbol{k}$ wektorly meýdanyň $x^2 + y^2 = 9$ silindr üstüň $x^2 + y^2 + z^2 = 25$ sfera bilen çäklenen böleginiň dasky tarapyndaky akymyny tapmaly.

2. Goý, S üst $x^2 + y^2 + z^2 = R^2$ sferanyň deňlemesi sfera koordinatalarynda $\theta = f_1(\varphi), \theta = f_2(\varphi)$ bolan konus üstler we $\varphi = \varphi_1$, $\varphi = \varphi_2$ ýarym tekizlikler bilen çäklenen bölegi bolsun.

Sfera koordinatalaryny girizeliň:

$x = R\cos\varphi\sin\theta$, $y = R\sin\varphi\sin\theta$, $z = R\cos\theta$,

$\varphi_1 \le \varphi \le \varphi_2$, $\theta_1 \le \theta \le \theta_2$.



12-nji surat

Onda dS element üçin $dS = R^2 \sin\theta\, d\theta\, d\varphi$ deňligi alarys. Bu halda $\boldsymbol{A}(P)$ wektorly meýdanyň sferanyň ýokardaky çäklendirilen böleginiň daşky tarapyndan akymy

$$K = R^2 \int_{\varphi_1}^{\varphi_2} d\varphi \int_{\theta_1}^{\theta_2} (\boldsymbol{A}(P), \boldsymbol{n}^0) \sin\theta\, d\theta$$

formula bilen hasaplanylýar. Bu ýerde

$$\boldsymbol{n}^0 = \frac{\text{grad}(x^2+y^2+z^2-R^2)}{\left|\text{grad}(x^2+y^2+z^2-R^2)\right|} = \frac{x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}}{R} .$$

12-nji mysal. $\boldsymbol{A}(P) = xz\boldsymbol{i} + yz\boldsymbol{j} + xy\boldsymbol{k}$ wektorly meýdanyň $x^2+y^2+z^2=R^2$ sferanyň birinji oktantadaky böleginiň daşky tarapyndaky akymyny tapmaly.

Çözülişi. Bu mysalda

$$\varphi_1 = 0, \varphi_2 = \frac{\pi}{2}, \theta_1 = 0, \theta_2 = \frac{\pi}{2}.$$

$$\boldsymbol{n}^0 = \frac{\text{grad}(x^2+y^2+z^2-R^2)}{\left|\text{grad}(x^2+y^2+z^2-R^2)\right|} = \frac{x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}}{R}.$$

$$(\boldsymbol{A}(P), \boldsymbol{n}^0) = \frac{3}{R}xyz.$$

Sferik koordinatalaryny girizeliň:
$x = R\cos\varphi\sin\theta$, $y = R\sin\varphi\sin\theta$, $z = R\cos\theta$.

Onda
$(\boldsymbol{A}(P), \boldsymbol{n}^0) = 3R^2\sin^2\theta\cos\theta\sin\varphi\cos\varphi.$

Akymy hasaplamagyň formulasyny ulanalyň:

$$K = 3R^4\int_0^{\frac{\pi}{2}}\sin\varphi\cos\varphi d\varphi\int_0^{\frac{\pi}{2}}\sin^3\theta\cos\theta d\vartheta = \frac{3}{4}R^4\int_0^{\frac{\pi}{2}}\sin\varphi\cos\varphi d\varphi = \frac{3}{8}R^4.$$

67. $\boldsymbol{A}(P) = (x-2y+1)\boldsymbol{i} + (2x+y-3z)\boldsymbol{j} + (3y+z)\boldsymbol{k}$ wektorly meýdanyň $x^2+y^2+z^2=1$ sferanyň birinji oktantadaky $x^2+y^2+z^2>1$ şerti kanagatlandyrýan bölegindäki akymyny tapmaly.

68. $\boldsymbol{A}(P) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ radius wektoryň $x^2+y^2+z^2=2$ sferanyň z=0, z=y tekizlikler bilen çäklenen böleginiň daşky tarapy boýunça akymyny tapmaly.

§ 5. Ýapyk üst boýunça wektoryň akymy

Goý, öz-özüni kesmeýän endigan ýapyk S üst bilen çäklenen V ýaýla berlen bolsun. $\boldsymbol{A}(P) = A_X(x, y, z)\boldsymbol{i} + A_y(x, y, z)\boldsymbol{j} + A_z(x, y, z)\boldsymbol{k}$

wektoryň $A_X(x, y, z)$, $A_y(x, y, z)$, $A_z(x, y, z)$ koordinatalary V ýaýlada kesgitlenen we özleriniň birinji tertipli hususy önümleri bilen ýapyk V ýaýlada üznüksiz bolsunlar. Onda $\boldsymbol{A}(P)$ wektoryň S üst boýunça akymy $\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$ funksiýadan alnan üçgat integrala deňdir:

$$K = \oiint_S (\boldsymbol{A}(P), \boldsymbol{n}^0)dS = \iiint_V \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) dV.$$

S üste geçirilen n normal üstüň daşky tarapy boýunça alnandyr. Bu formula Ostrogradskiý-Gaussyň formulasy diýilýär.

1-nji mysal. Ostrogradskiý-Gaussyň formulasyny ulanyp,

$\boldsymbol{A}(P) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ wektorly meýdanyň

$$x + y + z = 1, \ x = 0, \ y = 0, \ z = 0$$

tekizlikler bilen çäklenen piramidanyň daşky tarapy boýunça akymyny tapmaly.

Çözülişi. Ostrogradskiý-Gaussyň formulasynyň esasynda, alarys:

$$K = \oiint_S (\boldsymbol{A}(P), \boldsymbol{n}^0)dS = \iiint_V (1+1+1)dV = 3\iiint_V dxdydz =$$

$$= 3\int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz = 3\int_0^1 dx \int_0^{1-x} [z]_0^{1-x-y} = 3\int_0^1 \left[y - xy - \frac{y^2}{2} \right] dx =$$

$$= 3\int \left[1 - x - x(1-x) - \frac{(1-x)^2}{2} \right] dx = 3\frac{1}{6} = \frac{1}{2}.$$

2-nji mysal. $\boldsymbol{A}(P) = x^3\boldsymbol{i} + y^3\boldsymbol{j} + z^3\boldsymbol{k}$ wektorly meýdanyň $x^2 + y^2 + z^2 = x$ sferanyň daşky tarapy boýunça akymyny tapmaly.

Çözülişi. $\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}=3x^2+3y^2+3z^2$ deňlik esasynda wektoryň akymy tapmak üçin aşakdaky deňligi ýazyp bileris:

$$K=3\iiint\limits_V (x^2+y^2+z^2)dxdydz.$$

Sferik koordinatalara geçeliň:

$x=R\cos\varphi\sin\theta$, $y=R\sin\varphi\sin\theta$, $z=R\cos\theta$,

$$0\le\varphi\le\pi, -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}, 0\le R\le\sin\varphi\cos\theta.$$

Onda

$$K=3\int_0^{\pi}\sin\varphi d\varphi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{\sin\varphi\cos\theta}R^4dR=\frac{3}{5}\int_0^{\pi}\sin^6\varphi d\varphi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^5\theta d\theta=$$

$$=\frac{12}{5}\int_0^{\frac{\pi}{2}}\sin^6\varphi d\varphi\int_0^{\frac{\pi}{2}}\cos^5\theta d\theta=\frac{\pi}{5}.$$

69. $\boldsymbol{A}(P)=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$ radius-wektoryň $z=1-\sqrt{x^2+y^2}$, $0\le z\le 1$ üstüň daşky tarapy boýunça akymyny tapmaly.

70. $\boldsymbol{A}(P)=x^2\boldsymbol{i}+y^2\boldsymbol{j}+z^2\boldsymbol{k}$ wektorly meýdanyň $x^2+y^2+z^2=R^2$ z=0, (z>0) ýapyk üst boýunça akymyny tapmaly.

71. $\boldsymbol{A}(P)=(1+2x)\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$ wektorly meýdanyň $x^2+y^2=z^2$, z=4, $z\ge 0$ ýapyk üst boýunça akymyny tapmaly.

72. $\boldsymbol{A}(P)=x\boldsymbol{i}+xz\boldsymbol{j}+y\boldsymbol{k}$ wektorly meýdanyň $x^2+y^2=4-z$, z=0, $z\ge 0$ ýapyk üst boýunça akymyny tapmaly.

§ 6. Diwergensiýa

Berlen $\boldsymbol{A}(P)$ wektorly meýdanyň P nokadyny tutuşlygyna şu meýdanda ýerleşýän S üst bilen çäklendireliň. S üst boýunça wektor akymy hasaplap, ony şu üst bilen çäklendirilen D ýaýlasynyň V göwrümine bolan gatnaşygyna garalyň:

$$\frac{1}{V}\iint\limits_{(S)}(\boldsymbol{A}(P),\boldsymbol{n}^0)d\sigma. \qquad (1)$$

Bu gatnaşyk suwuklyklaryň tizlikleriniň meýdanynda birlik wagtda D ýaýlanyň birlik göwrümindäki suwuklygyň mukdarydyny aňladýar. Başgaça aýdanymyzda, bu gatnaşyga çeşmäniň ortaça göwrüm kuwwaty diýilýär. Eger S üstüň içinde akym noldan kiçi bolsa, onda gatnaşyk siňňidiň göwrüm kuwwatyny kesgitleýär.

D ýaýla gysylyp P nokada ýygnananda ýa-da $V \to 0$ bolanda

$$\lim_{D\to P}\frac{1}{V}\iint\limits_{S}(\boldsymbol{A}(P),\boldsymbol{n}^0)d\sigma$$

gatnaşygyň predelini tapalyň. Eger şu predel položitel bolsa P nokada çeşme, tersine bolanda bolsa siňňit diýilýär. Predeliň ululygy bolsa çeşmäniň ýa-da siňňidiň kuwwadyny aňladýar.Birinji halda P nokady çäklendirýän ujypsyz kiçi göwrümde suwuklyk döreýär, ikinji halda bolsa suwuklyk ýitip gidýär. Bu predele bolsa, wektor meýdanyň P nokatdaky diwergensiýasy ýa-da “dargamaklygy” diýilýär.

Kesgitleme. Ýapyk üst boýunça wektor akymyň, şol üst bilen çäklendirilen P nokady özünde saklaýan göwrüme bolan gatnaşygynyň, ähli üstüň gysylyp, P nokada ýygnanandaky predeline $\boldsymbol{A}(P)$ meýdanyň P nokatdaky diwergensiýasy diýilýär we div$\boldsymbol{A}(P)$ bilen belgilenýär.

Şunlukda,

$$\mathrm{div}\boldsymbol{A}(P)=\lim_{(S)\to P}\frac{\iint\limits_{S}(\boldsymbol{A}(P),\boldsymbol{n}^0)d\sigma}{V}.$$

Indi wektorly meýdanyň diwergensiýasynyň bardygyna we ony tapmaklygyň düzgünine garalyň. Onuň üçin A_x, A_y, A_z funksiýalaryň üznüksizligini we $\frac{\partial A_x}{\partial x}, \frac{\partial A_y}{\partial y}, \frac{\partial A_z}{\partial z}$ hususy önümleriň wektorly meýdanyň erkin nokadynda bar bolmagyny talap edeliň. Şu şertlerde aşakdaky teorema dogrudyr.

Teorema. Eger $\frac{\partial A_x}{\partial x}$, $\frac{\partial A_y}{\partial y}$, $\frac{\partial A_z}{\partial z}$ hususy önümleriň bahalary P nokatda alnan bolsa

$$\boldsymbol{A}(P) = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}$$

wektorly meýdanyň diwergensiýasy aşakdaky formula bilen tapylýar:

$$\operatorname{div}\boldsymbol{A}(P) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.$$

Subudy. Ostrogradskiý-Gaussyň formulasynyň esasynda wektoryň akymyny aşakdaky görnüşde ýazyp bileris:

$$\iint_S (\boldsymbol{A}(P), \boldsymbol{n}^0)ds = \iint_S (A_x \cos\alpha + A_y \cos\beta + A_z \cos\gamma)ds =$$

$$= \iiint_{(V)} \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) dxdydz.$$

Şu deňligiň sag bölegindäki üçgat integral orta baha baradaky teoremanyň esasynda integrirlenýän funksiýanyň (V) ýaýlanyň P_1 nokatdaky bahasynyň V göwrüme köpeldilmegine deňdir:

$$\iiint_{(V)} \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) dxdydz = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right)_{P_1} V.$$

Eger (V) ýaýla P nokada ýygnanýan bolsa, P_1 nokat hem P nokada ýygnanýandyr:

$$\operatorname{div}A(P) = \lim_{P_1 \to P} \frac{\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) \cdot V}{V} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.$$

Teorema subut edildi.

Diwergensiýanyň formulasyny ulanmak bilen, öňden belli bolan Ostrogradskiý-Gaussyň formulasyny aşakdaky wektor görnüşde ýazyp bileris:

$$\iint_{(S)} \boldsymbol{A_n}(P)ds = \iiint_{(V)} \operatorname{div} \boldsymbol{A}(P)dv.$$

Görnüşi ýaly, üstden akyp geçýän wektoryň akymy şu üst bilen çäklendirilen göwrüm boýunça diwergensiýadan alnan üçgat integrala deňdir.

Suwuklyklaryň akymynyň meýdanynda wektor görnüşli Ostrogradskiý-Gaussyň teoremasynyň ýönekeý manysy bardyr. Üst boýunça suwuklygyň akymy ähli çeşmeleriň kuwwatyna we akym çykalgasyna baglydyr. Başgaça aýdanymyzda, üst boýunça suwuklygyň akymy, seredýän ýaýlamyzdaky döreýän suwuklygyň mukdarydyr (eger akym çykalgasy siňňit, çeşmeden kuwwatly bolsa suwuklyk göwrümde gutarýar). Eger diwergensiýa ähli nokatlarda nola deň bolsa, onda erkin ýapyk üst boýunça akym nola deňdir.

1-nji mysal. Eger $\boldsymbol{A}(P)=a\boldsymbol{i}+b\boldsymbol{j}+c\boldsymbol{k}$ biratly meýdan bolsa, (bu ýerde a, b, c hemişelik sanlardyr) div$A(P)$ tapmaly.

Çözülişi. Şerte görä $A_x = a$, $A_y = b$, $A_z = c$. Onda,

$$\frac{\partial A_x}{\partial x} = 0, \frac{\partial A_y}{\partial y} = 0, \frac{\partial A_z}{\partial z} = 0$$

we

$$\mathrm{div}\boldsymbol{A}(P)=\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0.$$

2-nji mysal. Aýlanýan jisimiň çyzyklaýyn tizlikleriniň meýdanynyň diwergensiýasyny hasaplamaly.

Çözülişi. Hemişelik $\boldsymbol{\omega}$ burç tizligi bilen Oz okuň daşynda aýlanýan jisimiň çyzykly tizlikleriniň meýdany

$$\boldsymbol{\vartheta} = -\omega y\boldsymbol{i} + \omega x\boldsymbol{j}$$

funksiýa bilen kesgitlenýär. Bu ýerde $A_x = -\omega y$, $A_y = \omega x$.

Onda,

$$\frac{\partial A_x}{\partial x} = 0, \frac{\partial A_y}{\partial y} = 0,$$

$$\mathrm{div}\,\boldsymbol{\vartheta} = \frac{\partial(-\omega y)}{\partial x} + \frac{\partial(\omega x)}{\partial y} = 0.$$

3-nji mysal. $\boldsymbol{r}$ radius-wektor meýdanyň diwergensiýasyny hasaplamaly.

Çözülişi. Alarys:

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k},\, A_x = x,\, A_y = y,\, A_z = z.$$

Şunlukda,

$$\operatorname{div}\boldsymbol{\vartheta} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3.$$

Meýdanyň her bir nokady hemişelik kuwwatly çeşmedir. Wektor formaly Ostrogradskiý-Gaussyň teoremasynyň esasynda ýapyk ýaýla boýunça radius wektoryň akymy şu üst bilen çäklendirilen Ω ýaýlanyň üçeldilen göwrümine deňdir:

$$K = \iiint_{\Omega} 3dV = 3V.$$

§ 7. Diwergensiýanyň häsiýetleri

Diwergensiýa hasaplanan mahalynda onuň aşakdaky ýönekeý häsiýetlerinden peýdalanmak amatly bolýar.

Eger C_1 we C_2 skalýar hemişelikler bolsa

$$\operatorname{div}\left[C_1\boldsymbol{A}_1(P) + C_2\boldsymbol{A}_2(P)\right] = C_1\operatorname{div}\boldsymbol{A}_1(P) + C_2\operatorname{div}\boldsymbol{A}_2(P).$$

Hakykatdan-da, $\boldsymbol{A}_1(P)$ wektoryň proýeksiýalaryny A_{1x}, A_{1y}, A_{1z} bilen, $\boldsymbol{A}_2(P)$ wektoryň proýeksiýalaryny bolsa A_{2x}, A_{2y}, A_{2z} bilen belgiläp alýarys:

$$\left[C_1\boldsymbol{A}_1(P) + C_2\boldsymbol{A}_2(P)\right] = (C_1A_{1x} + C_2A_{2x})\boldsymbol{i} + (C_1A_{1y} + C_2A_{2y})\boldsymbol{j} +$$
$$+(C_1A_{1z} + C_2A_{2z})\boldsymbol{k}.$$

Onda,

$$\operatorname{div}\left[C_1\boldsymbol{A}_1(P) + C_2\boldsymbol{A}_2(P)\right] = C_1\frac{\partial A_{1x}}{\partial x} + C_2\frac{\partial A_{2x}}{\partial y} + C_1\frac{\partial A_{1y}}{\partial y} + C_2\frac{\partial A_{2y}}{\partial y} +$$
$$+C_1\frac{\partial A_{1z}}{\partial z} + C_2\frac{\partial A_{2z}}{\partial z} = C_1\operatorname{div}\boldsymbol{A}_1(P) + C_2\operatorname{div}\boldsymbol{A}_2(P).$$

Goý, $\boldsymbol{A}(P)$ wektorly meýdany kesgitleýän funksiýa, $u(P)$ skalýar funksiýa bolsun. Onda,

$$\operatorname{div}\left[\boldsymbol{u}(P)\boldsymbol{A}(P)\right] = u(P)\operatorname{div}\boldsymbol{A}(P) + \boldsymbol{A}(P)\operatorname{grad}u(P).$$

Hakykatdan-da,

$$u(P)\boldsymbol{A}(P) = uA_x\boldsymbol{i} + uA_y\boldsymbol{j} + uA_z\boldsymbol{k}.$$

Onda,

$$\operatorname{div}[u(P)\boldsymbol{A}(P)] = \frac{\partial(uA_x)}{\partial x} + \frac{\partial(uA_y)}{\partial y} + \frac{\partial(uA_z)}{\partial z} =$$

$$= u\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) + \left(A_x\frac{\partial u}{\partial x} + A_y\frac{\partial u}{\partial y} + A_z\frac{\partial u}{\partial z}\right).$$

$\operatorname{grad}u = \frac{\partial u}{\partial x}\boldsymbol{i} + \frac{\partial u}{\partial y}\boldsymbol{j} + \frac{\partial u}{\partial z}\boldsymbol{k}$ kesgitlemäniň esasynda deňligiň sag

böleginiň ikinji goşulyjysy $\boldsymbol{A}(P)$ wektor meýdanyň gradiýente skalýar köpeltmek hasylyna deňdir. Ol bolsa subut etmeli deňligimiziň dogrudygyny görkezýär.

Eger $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ bolsa,

$$\operatorname{div}\boldsymbol{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

deňligiň esasynda

$$\operatorname{div}u\boldsymbol{r} = 3u + \boldsymbol{r}\operatorname{grad}u$$

4-nji mysal. $\boldsymbol{A}(P) = \dfrac{-x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}}{\sqrt{x^2 + y^2}}$ wektorly meýdanyň $M(3,4,5)$

nokatda diwergensiýasyny tapmaly. $\boldsymbol{A}(P)$ wektoryň akymy $(x-3)^2 + (y-4)^2 + (z-5)^2 = \varepsilon^2$ sferada näçä deň bolar?

Çözülişi. $\operatorname{div}\boldsymbol{A}(P) = \dfrac{\partial A_x}{\partial x} + \dfrac{\partial A_y}{\partial y} + \dfrac{\partial A_z}{\partial z}$ formulany ulanalyň.

$$A_x = -\frac{x}{\sqrt{x^2 + y^2}}$$

$$A_y = \frac{y}{\sqrt{x^2 + y^2}},\ A_z = \frac{z}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial A_x}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right) = -\frac{y^2}{\sqrt{(x^2+y^2)^3}},$$

$$\frac{\partial A_y}{\partial y} = \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2}}\right) = \frac{x^2}{\sqrt{(x^2+y^2)^3}},$$

$$\frac{\partial A_z}{\partial z} = \frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2+y^2}}\right) = \frac{1}{\sqrt{(x^2+y^2)^3}}$$

Onda

$$\mathrm{div}\boldsymbol{A}(x,y,z) = \frac{x^2-y^2}{\sqrt{(x^2+y^2)^3}} + \frac{1}{\sqrt{x^2+y^2}},$$

$$\mathrm{div}\boldsymbol{A}(3,4,5) = \frac{9-16}{\sqrt{25^3}} + \frac{1}{\sqrt{25}} = \frac{18}{125}.$$

div $\boldsymbol{A}(x,y,z)$ ululygyň K akymyň dykyzlydygyna görä, onuň takyk bahasyny tapmak üçin ýapyk ýaýla hökmünde $V = (x-3)^2 + (y-4)^2 + (z-5)^2 \le \varepsilon^2$ tükeniksiz kiçi şary almak bilen,

$$\iint\limits_{(S)} (\boldsymbol{A}(P), \boldsymbol{n}^0)dS = \iiint\limits_{(V)} \mathrm{div}\boldsymbol{A}(P)dV$$

formuladan peýdalanmak bolar. Şaryň tükeniksiz kiçidigine görä $\mathrm{div}\boldsymbol{A}(x,y,z) \approx d\boldsymbol{A}(P)$ deňlik esasynda, K akymy hasaplap bileris:

$$K \approx \iiint\limits_{(V)} \mathrm{div}\boldsymbol{A}(P)dxdydz = \frac{18}{125}\cdot\frac{4}{3}\pi\varepsilon^3 = \frac{24}{125}\pi\varepsilon^3.$$

73. Eger $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ radius-wektor, $r = \sqrt{x^2+y^2+z^2}$ bolsa, $\mathrm{div}\left(\frac{x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}}{\sqrt{x^2+y^2+z^2}}\right)$ hasaplamaly.

74. $\boldsymbol{A}(P)=x^2\boldsymbol{i}+y^2\boldsymbol{j}+z^2\boldsymbol{k}$ wektorly meýdanyň diwergensiýasyny tapmaly.

75. $u=e^{x+y+z}$ funksiýanyň gradiýentiniň diwergensiýasyny tapmaly.

76. Eger $\boldsymbol{u}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}, \boldsymbol{v}=y\boldsymbol{i}+z\boldsymbol{j}+x\boldsymbol{k}$ bolsa, $\operatorname{div}[\boldsymbol{u}\times\boldsymbol{v}]-$ tapmaly.

77. $\operatorname{div}(\boldsymbol{u}\operatorname{grad}u)$ tapmaly.

78. $\operatorname{div}(\boldsymbol{u}\operatorname{grad}v)$ tapmaly.

79. Eger $r=\sqrt{x^2+y^2+z^2}$ bolsa, $\boldsymbol{A}(P)=f(r)\frac{r}{r}$ wektorly meýdanyň diwergensiýasyny tapmaly.

80. $\boldsymbol{A}(P)=x\boldsymbol{i}+y^2\boldsymbol{j}+z^3\boldsymbol{k}$ wektorly meýdanyň $M(-2,4,5)$ nokatdaky diwergensiýany tapmaly.

81. Formulany subut etmeli

$\oint\!\!\oint_S(\varphi\boldsymbol{A}(P),\boldsymbol{n}^0)dS=\iiint_V(\varphi\operatorname{div}\boldsymbol{A}(P))+(\boldsymbol{A}(P),\operatorname{grad}\varphi))dV,$ bu ýerde $\varphi=\varphi(x,y,z)$ we S üst bolsa V göwrümi çäklendirýär.

82. Eger $\boldsymbol{A}(P)=A_x\boldsymbol{i}+A_y\boldsymbol{j}+A_z\boldsymbol{k}$ wektor funksiýa, $\nabla\boldsymbol{u}=(\frac{\partial}{\partial x}\boldsymbol{i}+\frac{\partial}{\partial y}\boldsymbol{j}+\frac{\partial}{\partial z}\boldsymbol{k})u$ operator bolsa, $\nabla\boldsymbol{A}(P)=\operatorname{div}\boldsymbol{A}(P)$ deňligi subut etmeli.

83. Eger nokarlaýyn q zarýadyň elektrostatiki meýdany

$\boldsymbol{A}(P)=\frac{q}{4\pi e_0}\frac{r_1}{r^2},\ \boldsymbol{r}_1=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$ bolsa, $\operatorname{div}\boldsymbol{A}(P)$ hasaplamaly.

§ 8. Solenoidal meýdan

Kesgitleme. Eger $\operatorname{div}\boldsymbol{A}(P)=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}$ ululyk wektorly meýdanyň D ýaýlasynda nola deň bolsa, onda ol meýdana solenoidal meýdan diýilýär.

Diýmek, kesgitlemä görä solenoidal meýdanda çeşme hem, siňňit hem ýokdur.

Ostrogradskiý-Gaussyň teoremasynyň esasynda solenoidal meýdanda şu meýdana degişli erkin ýapyk S üst boýunça $\boldsymbol{A}(P)$ wektorly meýdanyň akymy nola deňdir:

$$\oiint (\boldsymbol{A}(P), \boldsymbol{n}^0) dS = 0.$$

$\mathrm{div}\boldsymbol{A}(P) = 0$ deňlemä, gidrodinamikada gysylmaýan suwuklyklaryň üznüksizlik deňlemesi diýilýär.

Bu halda haýsy hem bolsa ýapyk S üste girýän suwuklyk mukdary ondan çykýan suwuklygyň mukdaryna deňdir we onuň doly akymy nola deňdir.

Aşakdaky wektorly meýdanlaryň haýsylarynyň solenoidal meýdandygyny ýa-da solenoidal meýdan däldigini anyklamaly.

84. $\boldsymbol{A}(P) = x(z^2 - y^2)\boldsymbol{i} + y(x^2 - z^2)\boldsymbol{j} + z(y^2 - x^2)\boldsymbol{k}.$

85. $\boldsymbol{A}(P) = y^2\boldsymbol{i} - (x^2 + y^3)\boldsymbol{j} + z(3y^2 + 1)\boldsymbol{k}.$

86. $\boldsymbol{A}(P) = (1 + 2xy)\boldsymbol{i} - y^2 z\boldsymbol{j} + (z^2 y - 2zy + 1)\boldsymbol{k}.$

87. $\boldsymbol{A}(P) = (x - y)(y - z)\boldsymbol{i} + (y - z)(z - x)\boldsymbol{j} + (z - x)(x - y)\boldsymbol{k}.$

88. $\boldsymbol{A}(P) = (x^2 + y^2)(y - z)\boldsymbol{i} + (y^2 + z^2)(z - x)\boldsymbol{j} + (z^2 + x^2)(x - y)\boldsymbol{k}.$

89. $\boldsymbol{A}(P) = \left[x + f_1(y, z)\right]\boldsymbol{i} + \left[y + f_2(x, z)\right]\boldsymbol{j} + \left[z + f_3(x, y)\right]\boldsymbol{k}.$

§ 9. Wektorly meýdanda çyzykly integral

Goý, üznüksiz $\boldsymbol{A}(P)$ wektory bolan wektorly meýdan we položitel ugra ugrukdyrylan endigan L egri çyzyk berlen bolsun. L egri çyzyga geçirilen galtaşýanyň birlik wektoryny $\boldsymbol{n}^0$ bilen belgiläliň.

Kesgitleme $(\boldsymbol{A}(P), \boldsymbol{n}^0)$ skalýar köpeltmek hasylyndan L egri çyzyk boýunça alnan

$$\int_L (\boldsymbol{A}(P), \boldsymbol{n}^0) ds \qquad (1)$$

birinji görnüşli egriçyzykly integrala, $\boldsymbol{A}(P)$ wektordan alnan çyzykly integral diýilýär. Bu ýerde ds, L egri çyzygyň s dugasynyň differensialydyr.

Eger $\boldsymbol{r}=\boldsymbol{r}(M)$, L egri çyzygyň M nokadynyň radius wektory bolsa, onda $\boldsymbol{A}(P)$ wektoryň çyzykly integralyny aşakdaky görnüşde ýazyp bolýar:

$$\int\limits_L (\boldsymbol{A}(P), \boldsymbol{n}^0)ds = \int\limits_L (\boldsymbol{A}(P), d\boldsymbol{r}). \qquad (2)$$

Eger wektorly meýdanda gönüburçly dekart koordinatalar ulgamy girizilen bolsa, onda $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$,

$$\boldsymbol{A}(P) = A_x(x,y,z)\boldsymbol{i} + A_y(x,y,z)\boldsymbol{j} + A_z(x,y,z)\boldsymbol{k}$$

we (1) çyzykly integraly, ikinji görnüşli egriçyzykly integral bilen aňladylýar:

$$\int\limits_L (\boldsymbol{A}(P), d\boldsymbol{r}) = \int\limits_L A_x(x,y,z)dx + A_y(x,y,z)dy + A_z(x,y,z)dz.$$

Eger $\boldsymbol{A}=\boldsymbol{A}(P)$ wektorly meýdan güýç meýdany bolsa, onda (1) çyzykly integral bu meýdanda L egri boýunça edilen işi aňladýar.

Çyzykly integralyň birnäçe ýönekeý häsiýetlerini belläp geçeliň.

1. Eger m we n hemişelik sanlar bolsa, onda:

$$\int\limits_L (mA_1 + nA_2, d\boldsymbol{r}) = m\int\limits_L (A_1, d\boldsymbol{r}) + n\int\limits_L (A_2, d\boldsymbol{r});$$

2. $$\int\limits_{L_1+L_2} (\boldsymbol{A}_1, d\boldsymbol{r}) = \int\limits_{L_1} (\boldsymbol{A}_1, d\boldsymbol{r}) = \int\limits_{L_2} (\boldsymbol{A}_1, d\boldsymbol{r}).$$

3. Integrirlemegiň ugry üýtgedilse, integral alamatyny garşylykly alamata çalşyrýar:

$$\int\limits_{MN} (\boldsymbol{A}_1, d\boldsymbol{r}) = -\int\limits_{NM} (\boldsymbol{A}_1, d\boldsymbol{r}).$$

Wektorly meýdanda çyzykly integrallaryň hasaplanylyşynyň birnäçe görnüşine garalyň. Goý, L egri çyzyk özüniň parametrli deňlemesi bilen berlen bolsun:

$$x = x(t),\ y = y(t),\ z = z(t),\ t_0 \le t \le t_1.$$

Bu ýerde t parametriň $t=t_0$ bahasyna M nokat, $t=t_1$ bahasyna bolsa N nokat degişlidir. Eger $x(t)$, $y(t)$, $z(t)$ funksiýalar $[t_0, t_1]$ kesimde üznüksiz we üznüksiz önümlere eýe bolsalar, onda çyzykly integraly hasaplamak üçin, aşakdaky formula dogrudyr:

$$\int_L (\boldsymbol{A}(P), d\boldsymbol{r}) = \int_{MN} (\boldsymbol{A}(P), d\boldsymbol{r}) = \int_{t_0}^{t_1} \left\{ A_x \left[x(t), y(t), z(t) \right] x'(t) + \right.$$

$$\left. + A_y \left[x(t), y(t), z(t) \right] y'(t) + A_z \left[x(t), y(t), z(t) \right] z'(t) \right\} dt \cdot$$

Eger-de L egriniň deňlemesi $y = y(x)$, $z = z(x)$, $a \le x \le b$ görnüşde berlen bolsa, onda aşakdaky formula dogrudyr:

$$\int_{AB} (\boldsymbol{A}, d\boldsymbol{r}) = \int_a^b \left\{ A_x \left[x, y(x), z(x) \right] + A_y \left[x, y(x), z(x) \right] y'(x) + \right. \tag{3}$$

$$\left. + A_z \left[x, y(x), z(x) \right] z'(x) \right\} dx.$$

Şeýle hem L egri çyzygyň deňlemesi $x=x(y)$, $z=z(y)$, $y_0 \le y \le y_1$ ýa-da $x = x(z)$, $y = y(z)$, $z_0 \le z \le z_1$ formulalar bilen berlen bolsa, onda çyzykly integraly hasaplamak üçin (3) formula meňzeş bolan formulalary ýazyp bileris.

1-nji mysal. $\boldsymbol{A}(P)=r$ wektoryň $M(r_1)$ we $N(r_2)$ nokatlary birleşdirýän göni çyzyk boýunça çyzykly integralyny tapmaly.

Çözülişi. Kesgitlemäniň esasynda ýazyp bileris:

$$\int_{MN} (\boldsymbol{A}(P), d\boldsymbol{r}) = \int_{MN} (\boldsymbol{r}, d\boldsymbol{r}),$$

$d(\boldsymbol{r}, \boldsymbol{r}) = (d\boldsymbol{r}, \boldsymbol{r}) + (\boldsymbol{r}, d\boldsymbol{r}) = 2(\boldsymbol{r}, d\boldsymbol{r})$ deňligiň esasynda ýazyp bileris:

$$(\boldsymbol{r}, d\boldsymbol{r}) = \frac{1}{2} d(\boldsymbol{r}, \boldsymbol{r}) = \frac{1}{2} d(|r|^2) = \frac{1}{2} 2|r| d|r| = |r||dr|.$$

Onda

$$\int_{MN} (\boldsymbol{A}(P), d\boldsymbol{r}) = \frac{1}{2} \int_{MN} d(\boldsymbol{r}, \boldsymbol{r}) = \frac{1}{2} \int_{r_1}^{r_2} d(|r|^2) = \frac{|r_2|^2 - |r_1|^2}{2}.$$

2-nji mysal. $\boldsymbol{A}(P)=\boldsymbol{r}$ wektoryň, deňlemesi

$x = R\cos t,\ y = R\sin t,\ z = ht$, $0 \le t \le 2\pi$ bolan hyrly çyzygyň bir aýlawy boýunça çyzykly integralyny hasaplamaly.

Çözülişi. Kesgitlemäniň esasynda ýazyp bileris:

$$\int_L (\boldsymbol{A}(P), d\boldsymbol{r}) = \int_L xdx + ydy + zdz.$$

Hyrly çyzyk $x^2 + y^2 = R^2$ silindrde ýerleşen. Onuň başlangyç M nokadynda $t_0 = 0$, ahyrky N nokadynda $t_1 = 2\pi$.

$dx = -R\sin t,\ dy = R\cos t,\ dz = hdt.$

Onda

$$\int_L (\boldsymbol{A}(P), d\boldsymbol{r}) = \int_0^{2\pi} (-R\cos tR\sin t + r\sin tR\cos t + hth)dt =$$

$$= h^2 \int_0^{2\pi} tdt = h^2 \frac{t^2}{2}\Big|_0^{2\pi} = 2h^2\pi^2.$$

3-nji mysal. $\boldsymbol{F} = 2xy\boldsymbol{i} + x^2\boldsymbol{j}$ güýjiň täsiri bilen $x^2 + y^2 = 1$ töweregiň (1,0) we (0,1) nokatlaryny birleşdirýän dugasy boýunça ýerine ýetirilen işi hasaplamaly.

Çözülişi. Güýç meýdanynda ýerine ýetirilen iş aşakdaky çyzykly integrala deňdir:

$$\int_L (\boldsymbol{F}, d\boldsymbol{r}) = \int_L 2xydx + x^2dy,\ L: x^2 + y^2 = 1.$$

Töweregiň parametrli deňlemesini alyp ýazyp bileris:

$\left(x = \cos t,\ y = \sin t,\ 0 \le t \le 2\pi\right)$

$$\int_L (\boldsymbol{F}, d\boldsymbol{r}) = \int_L 2xydx + x^2dy = \int_0^{2\pi} \left[2\cos t\sin t(-\sin t) + \right.$$

$$\left. + \cos^2 t\cos t\right]dt = -2\int_0^{2\pi} \sin^2 t\cos tdt + \int_0^{2\pi} (1 - \sin^2 t)\cos tdt = 0.$$

4-nji mysal. $\boldsymbol{F} = y\boldsymbol{i} + x\boldsymbol{j} + (x + y + z)\boldsymbol{k}$ güýjüň täsiri bilen göni çyzygyň M(2,3,4) we N(3,4,5) nokatlaryny birleşdirýän kesimi boýunça ýerine ýetirilen işi hasaplamaly.

Çözülişi. Kesgitlemäniň esasynda ýazyp bileris:

$$\int_{MN} (\boldsymbol{F}, d\boldsymbol{r}) = \int_{MN} ydx + xdy + (x + y + z)dz.$$

Bu ýerden $y = x+1,\ z = x+2,\ dy = dx,\ dz = dx$. Onda

$$\int\limits_{MN} (\boldsymbol{F}, d\boldsymbol{r}) = \int\limits_{MN} ydx + xdy + (x+y+z)dz =$$

$$= \int_2^3 (x+1+x+x+x+1+x+2)dx = \int_2^3 (5x+4)dx = \frac{33}{2}.$$

Eger, r radius wektor bolsa, $A(P)$ wektoryň $M(r_M)$ we $M(r_N)$ nokatlary birleşdirýän çyzyk boýunça çyzykly integralyny tapmaly.

90. $\boldsymbol{A}(P) = \dfrac{r}{|r|}$,

91. $\boldsymbol{A}(P) = \dfrac{r}{|r|^2}$,

92. Deňligi subut etmeli

$$\int\limits_{MN} (\mathrm{grad}\, u, dr) = u(N) - u(M).$$

93. $\boldsymbol{A}(P) = \boldsymbol{i}z + x\boldsymbol{j} + y\boldsymbol{k}$ wektoryň deňlemesi $x=R\cos t$, $y=R\sin t$, $z=ht$, $0 \le t \le 2\pi$ bolan hyrly aýlawyň bir aýlawy boýunça alnan çyzykly integraly hasaplamaly.

94. $\boldsymbol{A}(P) = \dfrac{y^2\boldsymbol{i} - x^2\boldsymbol{j}}{\sqrt{x^2+y^2}}$ wektoryň deňlemesi $x=R\cos t$, $y=R\sin t$,

$0 \le t \le \pi$ bolan ýarym töwerek boýunça çyzykly integralyny hasaplamaly.

95. $\boldsymbol{A}(P) = \boldsymbol{i}y + z\boldsymbol{j} + x\boldsymbol{k}$ wektoryň deňlemesi $x=R\cos t$, $y=R\sin t$, $z=bt$, $0 \le t \le 2\pi$ bolan hyrly aýlawyň bir aýlawy boýunça çyzykly integralyny hasaplamaly.

96. $\boldsymbol{F} = 2xy\boldsymbol{i} + x^2\boldsymbol{j}$ güýjüň täsiri bilen $O(0,0)$ we $N(1,1)$ nokatlary birleşdirýän üç ugur boýunça ýerine ýetirilen işi hasaplamaly.

a) $y=x$ göni çyzyk;

b) $y = x^2$ parabola;

ç) $O(0,0)$, $N(1,0)$ we $N(1,0)$, $N(1,1)$ nokatlary birleşdirýän göni çyzyklaryň jemi.

97. $\boldsymbol{F}=(4x+y)\boldsymbol{i}+(x+4y)\boldsymbol{j}$ güýjüň täsiri bilen deňlemesi $y=x^4$ bolan egriniň $M(1,1)$, $N(-1,1)$ nokatlary birleşdirýän bölegi boýunça ýerine ýetirilen işi hasaplamaly.

98. $\boldsymbol{F}=(y^2-z^2)\boldsymbol{i}+2yz\boldsymbol{j}-x^2\boldsymbol{k}$ güýjüň täsiri bilen, deňlemesi $x=t,\ y=t^2, z=t^3, 0\le t\le 1$ görnüşde berlen L egri boýunca ýerine ýetirilen işi hasaplamaly.

99. $\boldsymbol{F}=x\boldsymbol{i}-y\boldsymbol{j}+(2x+y)\boldsymbol{k}$ güýjüň täsiri bilen deňlemesi $x=a\cos t,\ y=b\sin t,\ z=ct, 0\le t\le 2\pi$ bolan hyrly aýlawyň $A(a,0,0)$, $B(a,0,2\pi)$ nokatlaryny birleşdirýän bölegi boýunça ýerine ýetirilen işi hasaplamaly.

100. $\boldsymbol{F}=y\boldsymbol{i}-x\boldsymbol{j}$ güýjüň täsiri bilen $x^2+y^2=1$ töweregiň $A(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$ nokadyndan sagat diliniň ugry boýunça $B(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$ nokada çenli ýerine ýetirilen işi hasaplamaly.

101. $\boldsymbol{F}=\frac{\boldsymbol{i}}{y}+\frac{\boldsymbol{j}}{z}+\frac{\boldsymbol{k}}{x}$ güýjüň täsiri bilen göni çyzygyň $M(1,1,1)$ we $N(2,4,8)$ nokatlaryny birleşdirýän kesimi boýunça ýerine ýetirilen işi hasaplamaly.

§ 10. Wektorly meýdanyň köwlenmesi we onuň hasaplanylyşy

Kesgitleme. Üznüksiz $\boldsymbol{A}(P)$ wektorly meýdanyň $\boldsymbol{A}(P)$ wektorynyň položitel ugry bilen endigan ýapyk L egri çyzyga geçirilen galtaşýanyň $\boldsymbol{n}^0$ birlik wektoryna skalýar köpeltmek hasylandan, şu egri boýunça alnan çyzykly integrala wektor meýdanyň köwlenmesi diýilýär we aşakdaky görnüşde ýazylýar:

$$T=\oint_L(\boldsymbol{A}(P),\boldsymbol{n}^0)dl.$$

Bu ýerde $\oint$ belgi, integralyň ýapyk L egri boýunça alynýandygyny görkezýär.

Eger-de $\boldsymbol{A}(P)$ wektorly meýdan

$\boldsymbol{A}(P) = A_x(x,y,z)\boldsymbol{i} + A_y(x,y,z)\boldsymbol{j} + A_z(x,y,z)\boldsymbol{k}$

görnüşde, $d\boldsymbol{r} = \boldsymbol{i}dx + \boldsymbol{j}dy + \boldsymbol{k}dz$ bolsa, wektorly meýdanyň köwlenmesi

$$T = \oint_L A_x dx + A_y dy + A_z dz$$

formula bilen hasaplanylýar.

5-nji mysal. $\boldsymbol{A}(P) = -y\boldsymbol{i} + x\boldsymbol{j} + c\boldsymbol{k}$, $c = const$ wektorly meýdanyň $x^2 + y^2 = 1, z = 0$ töwerek boýunça köwlenmesini hasaplamaly.

Çözülişi. Köwlenmäniň kesgitlemesiniň esasynda

$$T = \oint_L (\boldsymbol{A}(P), \boldsymbol{n}^0)dl.$$

Töweregiň parametrik deňlemesini ýazalyň:

$$x = \cos t,\ y = \sin t,\ 0 \le t \le 2\pi .$$

Onda

$\boldsymbol{A}(P) = -\boldsymbol{i}\sin t + \boldsymbol{j}\cos t + c\boldsymbol{k}, n^0 = -\boldsymbol{i}\sin t + \boldsymbol{j}\cos t$ we

$$(\boldsymbol{A}(P), \boldsymbol{n}^0) = 1,\ dl = d\varphi.$$

$$T = \oint_L (\boldsymbol{A}(P), \boldsymbol{n}^0)dl = \int_0^{2\pi} d\varphi = 2\pi.$$

6-njy mysal. $x^2 + y^2 = 1$ silindr bilen $x + y + z = 1$ tekizligiň kesişmesinden alnan ýapyk L egri boýunça $\boldsymbol{A}(P) = xy\boldsymbol{i} + yz\boldsymbol{j} + xz\boldsymbol{k}$ wektorly meýdanyň köwlenmesini tapmaly.

Çözülişi.

$$T = \oint_L (\boldsymbol{A}(P), \boldsymbol{n}^0)dl = \int_L xydx + yzdy + xzdz.$$

L egri $x^2 + y^2 = 1$ silindr bilen $x + y + z = 1$ tekizligiň kesişmesindäki ellipsdir. Onuň parametrik deňlemesini tapalyň. Bu egriniň erkin nokadynyň xOy tekizlige bolan proýeksiýasy $x^2+y^2=1$ töwerekde ýatýandyr. $x=\cos t,\ y=\sin t$. Ellips $x+y+z=1$ tekizlikde ýatýandyr. Şonuň üçin $z=1-x-y$ ýa-da $z=1-\cos t-\sin t$. Onda L egriniň parametrik deňlemesi aşakdaky görnüşdedir:

$$\begin{cases} x=\cos t, \\ y=\sin t, \\ z=1-\cos t-\sin t, \end{cases} \quad 0\le t\le 2\pi$$

Bu ýerden $dx=-\sin t dt,\ dy=\cos t dt,\ dz=(\sin t-\cos t)dt$. Şunlukda,

$$T=\int_0^{2\pi}\left[-\cos t\sin^2 t+\sin t(1-\cos t-\sin t)\cos t+\right.$$

$$\left.+\cos t\,(1-\cos t-\sin t)(\sin t-\cos t)\right]dt=\int_0^{2\pi}(-3\sin^2 t\cos t+\sin 2t-$$

$$-\cos^2 t\sin t-\cos^2 t+\cos^3 t)dt=-\int_0^{2\pi}\cos^2 t dt=-\pi.$$

102. $\boldsymbol{A}(P)=-y\boldsymbol{i}+x\boldsymbol{j}+c\boldsymbol{k}, c=const$ wektorly meýdanyň $(x-2)^2+y^2=1,\ z=0$ töwerek boýunça köwlenmesini tapmaly.

103. $\boldsymbol{A}(P)=-y^3\boldsymbol{i}+x^3\boldsymbol{j}$ wektorly meýdanyň $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ ellips boýunça köwlenmesini hasaplamaly.

$\boldsymbol{A}(P)$ wektoryň L egri boýunça köwlenmesini tapmaly.

104. $\boldsymbol{A}(P)=(xz+y)\boldsymbol{i}+(yz-x)\boldsymbol{j}-(x^2+y^2)\boldsymbol{k} \quad L, \begin{cases} x^2+y^2=1 \\ z=3 \end{cases}$

105. $\boldsymbol{A}(P)=y^2\boldsymbol{i}+z^2\boldsymbol{j}+x^2\boldsymbol{k},\ L, \begin{cases} x^2+y^2+z^2=R^2 \\ x^2+y^2=Rx \end{cases},\ z\ge 0.$

§ 11. Wektorly meýdanyň rotory

Goý,

$$\boldsymbol{A}(P) = A_x(x,y,z)\boldsymbol{i} + A_y(x,y,z)\boldsymbol{j} + A_z(x,y,z)\boldsymbol{k}$$

wektorly meýdan berlen bolsun. $\boldsymbol{A}(P)$ wektoryň $A_x(x,y,z)$, $A_y(x,y,z)$, $A_z(x,y,z)$ koordinatalary we olaryň birinji tertipli hususy önümleri wektorly meýdanda üznüksiz funksiýalar bolsunlar.

1-nji kesgitleme. $\boldsymbol{A}(P)$ wektoryň rotory rot $\boldsymbol{A}(P)$ belgi bilen belgilenýär we

$$\operatorname{rot}\boldsymbol{A}(P) = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\boldsymbol{k} \qquad (1)$$

deňlik bilen ýa-da ýatlamaga amatly bolan

$$\operatorname{rot}\boldsymbol{A}(P) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \qquad (2)$$

deňlik bilen kesgitlenýär.

Bu kesgitleýji köplenç birinji setir boýunça dargadylyp açylýar. Şonda ikinji setiriň agzalarynyň üçünji setiriň agzalaryna köpeltmek hasylyna, differensirlemek hökmünde garaýarlar. Mysal üçin,

$$\frac{\partial}{\partial y}A_z = \frac{\partial A_z}{\partial y}.$$

Eger wektorly meýdanyň G ýaýlasynda rot $\boldsymbol{A}(P)$=0 bolsa, onda ol ýaýla potensial meýdan diýilýär.

1-nji mysal. Eger gaty jisim hemişelik $\boldsymbol{\omega}$ burçlaýyn tizlik bilen hereket edýän bolsa, jisimiň çyzyklaýyn tizlikleriniň meýdanynyň rotoryny tapmaly.

Çözülişi. Belli bolşy ýaly, jisimiň çyzyklaýyn tizlikleriniň meýdany (*§2.1, 1-nji mysal*) aşakdaky görnüşdedir:

$$\boldsymbol{v} = [\boldsymbol{\omega} \times \boldsymbol{r}] = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = x\omega \boldsymbol{j} - y\omega \boldsymbol{i}.$$

Onda

$$\operatorname{rot} \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y\omega & x\omega & 0 \end{vmatrix} = \boldsymbol{i} \cdot 0 + \boldsymbol{j} \cdot 0 + 2\omega \boldsymbol{k} = 2\omega k.$$

Şunlukda, rot$v$ hemişelik wektor bolup, onuň moduly, ikeldilen burç tizligine deňdir: $|\operatorname{rot} v| = 2\omega$.

2-nji mysal. $\boldsymbol{A}(P) = (x^2 + y^2)\boldsymbol{i} + (y^2 + z^2)\boldsymbol{j} + (z^2 + x^2)\boldsymbol{k}$ wektoryň rotoryny tapmaly.

Çözülişi. (2) formulany ulanyp, alýarys:

$$\operatorname{rot} \boldsymbol{A}(P) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 + y^2 & y^2 + z^2 & z^2 + x^2 \end{vmatrix} =$$

$$= \boldsymbol{i} \begin{vmatrix} \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 + z^2 & z^2 + x^2 \end{vmatrix} - \boldsymbol{j} \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial z} \\ x^2 + y^2 & z^2 + x^2 \end{vmatrix} +$$

$$+ \boldsymbol{k} \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ x^2 + y^2 & y^2 + z^2 \end{vmatrix} =$$

$$= \boldsymbol{i}(0 - 2z) - \boldsymbol{j}(2x - 0) + \boldsymbol{k}(0 - 2z) = -2(z\boldsymbol{i} + x\boldsymbol{j} + y\boldsymbol{k}).$$

3-nji mysal. rot*(*grad*u)*=0 bolýandygyny görkezmeli.

Çözülişi.

$$\operatorname{grad} u = \frac{\partial u}{\partial x} \boldsymbol{i} + \frac{\partial u}{\partial y} \boldsymbol{j} + \frac{\partial u}{\partial z} \boldsymbol{k}.$$

Bu ýerde $A_x = \dfrac{\partial u}{\partial x}$, $A_y = \dfrac{\partial u}{\partial y}$, $A_z = \dfrac{\partial u}{\partial z}$. Onda

$$\text{rot}(\text{grad}u) = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\boldsymbol{i} + *\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\boldsymbol{k} =$$

$$= \left(\frac{\partial^2 u}{\partial y \partial z} - \frac{\partial^2 u}{\partial y \partial z}\right)\boldsymbol{i} + \left(\frac{\partial^2 u}{\partial x \partial z} - \frac{\partial^2 u}{\partial z \partial x}\right)\boldsymbol{j} + \left(\frac{\partial^2 u}{\partial y \partial x} - \frac{\partial^2 u}{\partial x \partial y}\right)\boldsymbol{k} = 0.$$

Wektorlaryň rotorlaryny tapmaly.

106. $\boldsymbol{A}(P) = (x+z)\boldsymbol{i} + (y+z)\boldsymbol{j} + (x^2+z)\boldsymbol{k}$.

107. $\boldsymbol{A}(P) = z^3\boldsymbol{i} + y^3\boldsymbol{j} + x^3\boldsymbol{k}$.

108. $\boldsymbol{A}(P) = \dfrac{1}{2}(-y^2\boldsymbol{i} + x^2\boldsymbol{j})$.

109. $\boldsymbol{A}(P) = \dfrac{y}{x}\boldsymbol{i} + \dfrac{z}{y}\boldsymbol{j} + \dfrac{x}{z}\boldsymbol{k}$ wektorly meýdanyň $M(-1,-1,-1)$ nokatda rotoryny tapmaly.

110. $\boldsymbol{A}(P) = z^2\boldsymbol{i} + x^2\boldsymbol{j} + y^2\boldsymbol{k}$ wektorly meýdanyň $M(1,2,3)$ nokatda rotoryny tapmaly.

111. Eger $\boldsymbol{A}(P)$ we $\boldsymbol{B}(P)$ wektorly meýdanlar bolsa, onda

a) $\text{rot}(\boldsymbol{A} \pm \boldsymbol{B}) = \text{rot}\,\boldsymbol{A} \pm \text{rot}\,\boldsymbol{B},$ b) $\text{rot}(\lambda\boldsymbol{A}) = \lambda\,\text{rot}\,\boldsymbol{A},$

λ – hemişelik san, bolýandygyny görkezmeli.

112. Eger $\boldsymbol{A}(P)$ wektorly meýdan, $u(P)$ skalýar meýdan bolsa, onda

$$\text{rot}(u\boldsymbol{A}) = u\,\text{rot}\,\boldsymbol{A} + \text{grad}u \cdot \boldsymbol{A}$$

deňligi subut etmeli.

113. Eger $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, $a = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ bolsa $\text{rot}(ra)r$ tapmaly.

114. Eger $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, $a = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$, $b = \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$ bolsa $\text{rot}(ra)b$ tapmaly.

115. Eger $\boldsymbol{A}(P)$ wektoryň koordinatalarynyň ikinji tertipli üznüksiz önümleri bar bolsa, onda $\text{div}\,\text{rot}\,\boldsymbol{A}(P) = 0$ bolýandygyny görkezmeli.

116. $\text{div}\left[\boldsymbol{A}(P)\boldsymbol{B}(P)\right] = (\boldsymbol{B}(P), rot\boldsymbol{A}(P)) - (\boldsymbol{A}(P), rot\boldsymbol{B}(P))$ deňligi subut etmeli.

III BÖLÜM
POTENSIAL MEÝDAN

§ 1. Meýdanyň potensiallyk nyşany

Goý, giňişligiň V ýaýlasynda

$$\boldsymbol{A}(P) = A_x(x,y,z)\boldsymbol{i} + A_y(x,y,z)\boldsymbol{j} + A_z(x,y,z)\boldsymbol{k}$$

wektorly meýdan berlen bolsun. Eger tutuş V ýaýlada

$$\boldsymbol{A}(P) = \operatorname{grad}\varphi(P) \tag{1}$$

deňligi kanagatlandyrýan $\varphi(x,y,z)$ funksiýa bar bolsa, $\boldsymbol{A}(P)$ wektorly meýdana potensial meýdan, $\varphi(P) = \varphi(x,y,z)$ funksiýa bolsa, wektorly meýdanyň potensialy diýilýär.

(1) deňlik aşakdaky üç deňlik bilen deňgüýçlüdir.

$$A_x(x,y,z) = \frac{\partial\varphi}{\partial x},\ A_y(x,y,z) = \frac{\partial\varphi}{\partial y},\ A_z(x,y,z) = \frac{\partial\varphi}{\partial z}. \tag{2}$$

1-nji mysal. Goý, koordinatalar başlangyjynda m massaly O nokat ýerleşen bolsun. Eger $M(x,y,z)$ nokatda ýerleşen m_1 massaly nokady, O nokat ululygy $\frac{a}{r^2}, a = \gamma m m_1$, bolan $\boldsymbol{F}$ güýç bilen özüne dartýan bolsa, onda

$$\boldsymbol{F} = \operatorname{grad}\frac{a}{r}$$

bolýandygyny görkezmeli.

Çözülişi. Bu ýerde $\boldsymbol{F}$ güýç O nokada tarap ugrukdyrylandyr. Şonuň üçin $\boldsymbol{F} = -\lambda\boldsymbol{r},\ \lambda > 0$ we $|F| = \lambda|r| = \lambda r$ bolar. Beýleki tarapdan $|F| = \frac{a}{r^2}$. F-iň iki bahasyny deňläp, alarys: $\frac{a}{r^2} = \lambda r$ ýa-da $\lambda = \frac{a}{r^3}$.

λ-nyň tapylan bahasyny ýerine goýup alarys: $\boldsymbol{F} = -\frac{a}{r^3}\boldsymbol{r}$. Bu ýerde

$$A_x(x,y,z) = -\frac{ax}{r^3},\ A_y(x,y,z) = -\frac{ay}{r^3},\ A_z(x,y,z) = -\frac{az}{r^3},$$

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{\partial r}{\partial x} = -\frac{x}{r^3},\ \frac{\partial}{\partial y}\left(\frac{1}{r}\right) = -\frac{y}{r^3},\ \frac{\partial}{\partial z}\left(\frac{1}{r}\right) = -\frac{z}{r^3}.$$

Onda $\varphi(x,y,z) = \frac{a}{r} = \frac{a}{\sqrt{x^2+y^2+z^2}}$ funksiýa $\boldsymbol{F}$ güýç meýdanynyň potensialydyr: $\boldsymbol{F} = \mathrm{grad}\,\frac{a}{r}$.

Teorema. Giňişligiň

$$\boldsymbol{A}(P) = A_x(x,y,z)\boldsymbol{i} + A_y(x,y,z)\boldsymbol{j} + A_z(x,y,z)\boldsymbol{k}$$

wektorly V meýdanynyň potensial meýdan bolmaklygy üçin rot$A(P)$=0 şertiň ýerine ýetmekligi zerur we ýeterlikdir.

Başgaça aýdanymyzda wektorly meýdanyň towsuz bolmaklygy üçin onuň potensial meýdan bolmaklygy zerur we ýeterlikdir.

$$\boldsymbol{A}(P) = A_x(x,y,z)\boldsymbol{i} + A_y(x,y,z)\boldsymbol{j} + A_z(x,y,z)\boldsymbol{k}$$

wektorly meýdanyň $\varphi(x,y,z)$ potensialy

$$\varphi(x,y,z) = \int_{(x_0,y_0,z_0)}^{(x_1,y_1,z_1)} A_x dx + A_y dy + A_z dz$$

formula bilen tapylýar.

2-nji mysal. $\boldsymbol{A}(P) = (2xy+z^2)\boldsymbol{i} + (2yz+x^2)\boldsymbol{j} + (2xz+y^2)\boldsymbol{k}$ wektorly meýdanyň potensial meýdandygyny görkezmeli.

Çözülişi.

$\boldsymbol{A}(P)$ wektoryň

$$A_x = 2xy+z^2,\ A_y = 2yz+x^2,\ A_z = 2xz+y^2$$

koordinatalary differensirlenýän funksiýalardyrlar. Onda

$$\operatorname{rot}\boldsymbol{A}(P)=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 2xy+z^2 & 2yz+x^2 & 2xz+y^2\end{vmatrix}=$$

$$=\boldsymbol{i}\left(\frac{\partial}{\partial y}(2xz+y^2)-\frac{\partial}{\partial z}(2yz+x^2)\right)+\boldsymbol{j}\left(\frac{\partial}{\partial z}(2xy+z^2)-\frac{\partial}{\partial x}(2xz+y^2)\right)+$$

$$+\boldsymbol{k}\left(\frac{\partial}{\partial x}(2yz+x^2)-\frac{\partial}{\partial y}(2xy+z^2)\right)=0.$$

Diýmek, $\boldsymbol{A}(P)$ wektorly meýdan potensial meýdandyr.

3-nji mysal.

$\boldsymbol{A}(P)=yz(2x+y+z)\boldsymbol{i}+xz(x+2y+z)\boldsymbol{j}+xy(x+y+2z)\boldsymbol{k}$

wektorly meýdanyň potensial meýdandygyny görkezmeli we onuň potensialyny tapmaly.

Çözülişi.

$A_x=yz(2x+y+z),\ A_y=xz(x+2y+z),\ A_z=xy(x+y+2z),$

$$\operatorname{rot}\boldsymbol{A}(P)=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ yz(2x+y+z) & xz(x+2y+z) & xy(x+y+2z)\end{vmatrix}=$$

$$=\boldsymbol{i}\left(\frac{\partial}{\partial y}(xy(x+y+2z)\right)-\frac{\partial}{\partial z}\big(xz(x+2y+z)\big)+\boldsymbol{j}\left(\frac{\partial}{\partial z}yz(2x+y+z)-\right.$$

$$\left.-\frac{\partial}{\partial x}(xy(x+y+2z)\right)+\boldsymbol{k}\left(\frac{\partial}{\partial x}(xz(x+2y+z)-\frac{\partial}{\partial y}(yz(2x+y+z)\right)=0.$$

Wektorly meýdanyň potensial meýdandygynyň esasynda

$$\boldsymbol{A}(P)=\operatorname{grad}\varphi=\boldsymbol{i}\frac{\partial\varphi}{\partial x}+\boldsymbol{j}\frac{\partial\varphi}{\partial y}+\boldsymbol{k}\frac{\partial\varphi}{\partial z},$$

$$\frac{\partial\varphi}{\partial x}=yz(2x+y+z),\ \frac{\partial\varphi}{\partial y}=xz(x+2y+z),\ \frac{\partial\varphi}{\partial z}=xy(x+y+2z)$$

deňliklerden alarys:

$$\varphi(x,y,z)=xyz(x+y+z)+\varphi(y,z),$$

$$\frac{\partial\varphi}{\partial y}=xz(x+2y+z)=xz\left(x+2y+z\right)+\frac{\partial\varphi}{\partial y}.$$

Bu ýerden $\frac{\partial\varphi}{\partial y}=0$, $\varphi=\Phi(z)$.

$$\frac{\partial\varphi}{\partial z}=xy(x+y+2z)=xy(x+y+2z)+\Phi'(z)$$

deňlikden $\Phi'(z)=0,\ \Phi(z)=c, c=const$.

Şunlukda,

$$\varphi(x,y,z)=xyz(x+y+z)+c,\ c=const.$$

Aşakdaky wektorly meýdanlaryň potensial ýa-da potensial meýdan däldigini anyklamaly.

117. $\boldsymbol{A}(P)=x^2\boldsymbol{i}+y^2\boldsymbol{j}+z^2\boldsymbol{k}$.

118. $\boldsymbol{A}(P)=xz\boldsymbol{i}+zy\boldsymbol{j}+xy\boldsymbol{k}$.

119. $\boldsymbol{A}(P)=y^2\boldsymbol{i}+2xy\boldsymbol{j}+z\boldsymbol{k}$ wektorly meýdanyň potensial meýdandygyny görkezmeli we onuň potensialyny tapmaly.

Aşakdaky wektorly meýdanlaryň potensial meýdandygyny barlamaly. Eger potensial meýdan bolsa, onuň potensialyny tapmaly.

120.1. $\boldsymbol{A}_1(P)=(y+z)\boldsymbol{i}+(z+x)\boldsymbol{j}+(x+y)\boldsymbol{k}$.

2. $\boldsymbol{A}_2(P)=f_1(x)\boldsymbol{i}+f_2(x)\boldsymbol{j}+f_3(x)\boldsymbol{k}$.

3. $\boldsymbol{A}_3(P)=x\boldsymbol{i}+y\boldsymbol{j}+y\boldsymbol{k}$.

4. $\boldsymbol{A}_4(P)=z\boldsymbol{i}+x\boldsymbol{j}+y\boldsymbol{k}$.

121. $\boldsymbol{A}(P)=-\frac{y}{x^2+y^2}\boldsymbol{i}+\frac{x}{x^2+y^2}\boldsymbol{j}+2\boldsymbol{k}$.

122. $\boldsymbol{A}(P)=yz\cos xy\cdot\boldsymbol{i}+xz\cos xy\cdot\boldsymbol{j}+\sin xy\cdot\boldsymbol{k}$.

123. $\boldsymbol{A}(P)=\ln(1+z^2)\boldsymbol{i}+\ln(1+x^2)\boldsymbol{j}+xz\boldsymbol{k}$.

124. $\boldsymbol{A}(P)=\left(\frac{z}{x^2}+\frac{1}{y}\right)\boldsymbol{i}+\left(\frac{x}{y^2}+\frac{1}{z}\right)\boldsymbol{j}+\left(\frac{y}{z^2}+\frac{1}{x}\right)\boldsymbol{k}$.

125. $\boldsymbol{H}=\frac{2I}{r^2}(-y\boldsymbol{i}+x\boldsymbol{j}),\ r^2=x^2+y^2,\ r\neq 0$.

§ 2. Potensial meýdanda çyzykly integralyň hasaplanylyşy

$\boldsymbol{A}(P)$ potensial meýdanda $M(x_1,y_1,z_1)$ nokatdan $N(x_2,y_2,z_2)$ nokada çenli çyzykly integral $\varphi(P)$ potensialyň M we N nokatlardaky bahalarynyň tapawudyna deňdir:

$$\int_M^N (\boldsymbol{A}(P),dr)=\varphi(N)-\varphi(M). \tag{1}$$

Güýç meýdanynda material nokat M nokatdan N nokada geçende bitirilýän iş diňe M we N nokatlara baglydyr hem-de haýsy ýol bilen M nokatdan N nokada barlanyna bagly däldir. Has takygy, ol bitirilen iş potensial funksiýanyň M we N nokatdaky bahalarynyň tapawudyna, ýagny $\varphi(N)-\varphi(M)$ tapawuda deňdir.

1-nji mysal. $\boldsymbol{A}(P)=yz\boldsymbol{i}+zx\boldsymbol{j}+xy\boldsymbol{k}$ wektorly meýdanyň $M(1,1,1)$

nokadyndan $N(2,3,4)$ nokadyna çenli aralykda çyzykly integralyny hasaplamaly.

Çözülişi. Bu meýdanyň potensial meýdandygyny görkezeliň.

$A_x=yz, A_y=zx, A_{z=xy}$

$$\operatorname{rot}\boldsymbol{A}(P)=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ yz & zx & xy\end{vmatrix}=\boldsymbol{i}\begin{vmatrix}\frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ zx & xy\end{vmatrix}+\boldsymbol{j}\begin{vmatrix}\frac{\partial}{\partial z} & \frac{\partial}{\partial x}\\ xy & yz\end{vmatrix}+\boldsymbol{k}\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\\ yz & zx\end{vmatrix}=0.$$

Wektorly meýdanyň potensialyny tapalyň:

$$\boldsymbol{A}(P)=\operatorname{grad}\varphi=\frac{\partial\varphi}{\partial x}\boldsymbol{i}+\frac{\partial\varphi}{\partial y}\boldsymbol{j}+\frac{\partial\varphi}{\partial z}\boldsymbol{k}.$$

$$\frac{\partial\varphi}{\partial x}=yz,\quad \frac{\partial\varphi}{\partial y}=zx,\quad \frac{\partial\varphi}{\partial z}=xy$$

deňliklerden alýarys:

$$\frac{\partial\varphi}{\partial y}=zx=xz+\frac{\partial\varphi}{\partial y},\ \frac{\partial\varphi}{\partial y}=0,\ \varphi=\Phi(z),\ \varphi(x,y,z)=xyz+\Phi(z)$$

$$\frac{\partial\varphi}{\partial z}=xy+\Phi'(z)=xy,\quad \Phi'(z)=0,\quad \Phi(z)=0\,.$$

Şunlukda, $\varphi(x,y,z)=xyz+c$. (1) fopmulany ulanyp, taparys:

$$\int\limits_M^N(\boldsymbol{A}(P),d r)=\varphi(2,3,4)-\varphi(1.1.1)=24-1=23.$$

126. $\boldsymbol{A}(P)=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$ wektorly meýdanyň M(–1,0,3) nokadyndan N(2,–1,0) nokadyna çenli aralykda çyzykly integralyny hasaplamaly.

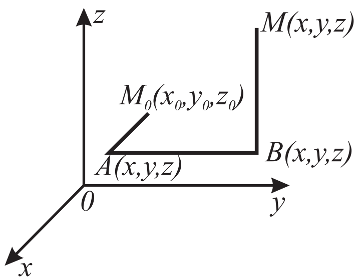
127. $\boldsymbol{A}(P)=(y+z)\boldsymbol{i}+(x+z)\boldsymbol{j}+(x+y)\boldsymbol{k}$ wektorly meýdanyň M(–1,0,5) nokatdan N(2,1,4) nokadyna çenli aralykda çyzykly integralyny hasaplamaly.

§ 3. Potensialyň dekart koordinatalar ulgamynda hasaplanylyşy

Goý, $\boldsymbol{A}(P)=A_x(x,y,z)\boldsymbol{i}+A_y(x,y,z)\boldsymbol{j}+A_z(x,y,z)\boldsymbol{k}$ wektorly potensial meýdan berlen bolsun. Bu meýdanyň $\varphi(x,y,z)$ potensialyny tapmak üçin

$$\varphi(x,y,z)=\int\limits_{(x_0,y_0,z_0)}^{(x_1,y_1,z_1)}A_x dx+A_y dy+A_z dz \qquad (1)$$

formuladan peýdalanmak amatlydyr. Onuň üçin başlangyç $M(x_0,y_0,z_0)$



13-nji surat

nokady *M(x,y,z)* nokady döwük çyzyklar arkaly birikdireliň. Bu suratda $M_0A\|Ox$, $AB\|Oy$, $BM\|Oz$ bolýanlygy sebäpli (1) formulany potensialy hasaplamak üçin amatly görnüşde ýazyp bileris:

$$\varphi(x,y,z)=\int_{x_0}^{x}A_x(x,y_0,z_0)dx+\int_{y_0}^{y}A_y(x,y,z_0)dy+\int_{z_0}^{z}A_z(x,y,z)dz. \quad (2)$$

1-nji mysal. $\boldsymbol{A}(P)=2xyz\boldsymbol{i}+x^2z\boldsymbol{j}+x^2y\boldsymbol{k}$ wektorly meýdanyň potensial meýdandygyny görkezmeli we onuň potensialyny tapmaly.

Çözülişi.

$$\operatorname{rot}\boldsymbol{A}(P)=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 2xyz & x^2z & x^2y\end{vmatrix}=\boldsymbol{i}(x^2-x^2)+\boldsymbol{j}(2xy-2xy)+\boldsymbol{k}(2xz-2xz)=0.$$

Meýdanyň potensialyny (2) formuladan peýdalanyp taparys. Onuň üçin başlangyç nokady $O(0,0,0)$ nokat diýip kabul edeliň. Onda

$$\varphi(x,y,z)=\int_0^x 0dx+\int_0^y 0dy+\int_0^z 2xydz=x^{2x}yz+C.$$

2-nji mysal. $\boldsymbol{A}(P)=(y+z)\boldsymbol{i}+(x+z)\boldsymbol{j}+(x+y)\boldsymbol{k}$ wektorly meýdanyň potensial meýdandygyny görkezmeli we onuň potensialyny tapmaly.

Çözülişi.

$$\operatorname{rot} \boldsymbol{A}(P) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y+z & x+z & x+y \end{vmatrix} = \boldsymbol{i}(1-1) + \boldsymbol{j}(0-0) + \boldsymbol{k}(1-1) = 0.$$

Diýmek, wektorly meýdan potensial meýdandyr. $\varphi(x, y, z)$ funksiýanyň doly differensialynyň kesgitlemesiniň esasynda yazyp bileris:

$$d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz\ .$$

Bu ýerde $\dfrac{\partial\varphi}{\partial x} = y+z, \dfrac{\partial\varphi}{\partial y} = x+z, \dfrac{\partial\varphi}{\partial z} = x+y$ goýup, alarys:

$$d\varphi = (y+z)dx + (x+z)dy + (x+y)dz.$$

Bu deňligi özgerdip ýazalyň:

$$d\varphi = (ydx + xdy) + (zdx + xdz) + (ydz + zdy) =$$

$$= d(xy) + d(xz) + d(yz) = d(xy + xz + yz).$$

Onda

$\varphi(x, y, z) = xy + xz + yz + C\ .$

Wektorly meýdanlaryň potensial meýdandygyny görkezmeli we onuň potensialyny tapmaly.

128. $\boldsymbol{A}(P) = (yz+1)\boldsymbol{i} + xz\boldsymbol{j} + xy\boldsymbol{k}.$

129. $\boldsymbol{A}(P) = (2xy+z)\boldsymbol{i} + (x^2-2y)\boldsymbol{j} + x\boldsymbol{k}.$

130. $\boldsymbol{A}(P) = \dfrac{\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}}{x+y+z}.$

131. $\boldsymbol{A}(P) = \dfrac{yz\boldsymbol{i} + xz\boldsymbol{j} + xy\boldsymbol{k}}{1+x^2y^2z^2}.$

132. $\boldsymbol{A}(P) = \alpha\boldsymbol{i} + \beta\boldsymbol{j} + \gamma k, \alpha, \beta, \gamma -$ hemişelik sanlar.

133. $\boldsymbol{A}(P) = y\boldsymbol{i} + x\boldsymbol{j} + e^z\boldsymbol{k}.$

134. $\boldsymbol{A}(P) = e^x \sin y \cdot \boldsymbol{i} + e^x \cos y \cdot \boldsymbol{j} + \boldsymbol{k}.$

IV BÖLÜM

GAMILTON OPERATORY. IKINJI TERTIPLI DIFFERENSIAL OPERATORLAR

§ 1. Gamilton operatory

Wektor derňewiň esasy düşünjeleri bolan gradiýenti, diwergensiýany we rotory, ∇ belgi bilen belgilenýän aşakdaky wektor bilen aňlatmak amatlydyr.

$$\nabla = \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}. \qquad (1)$$

Bu operator differensirlemegi wektora skalýar köpeltmek hasyllarynyň jeminden ybaratdyr. Şeýle-de, $\frac{\partial}{\partial x}$ differensirlemek amalynyň $u(x,y,z)$ funksiýa köpeltmek hasylyna $\frac{\partial u}{\partial x}$ hususy önüm hökmünde garaýarys.

1. ∇ (nabla) wektoryň $u(P)$ skalýar funksiýa skalýar köpeltmek hasyly bu funksiýanyň gradiýentidir:

$$\nabla u = \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}k\right)u = \frac{\partial u}{\partial x}\boldsymbol{i} + \frac{\partial u}{\partial y}\boldsymbol{j} + \frac{\partial u}{\partial z}\boldsymbol{k} = \operatorname{grad} u.$$

2. ∇ wektoryň

$$\boldsymbol{A}(P) = A_x(x,y,z)\boldsymbol{i} + A_y(x,y,z)\boldsymbol{j} + A_z(x,y,z)\boldsymbol{k}$$

wektor funksiýa skalýar köpeltmek hasyly bu funksiýanyň diwergensiýasydyr:

$$\nabla \cdot \boldsymbol{A}(P) = \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k} \right)\left(A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k} \right) =$$

$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \operatorname{div} \boldsymbol{A}(P).$$

3. ∇ wektoryň $\boldsymbol{A}(P)$ wektor funksiýa wektor köpeltmek hasyly bu funksiýanyň rotoryna deňdir:

$$\nabla \times \boldsymbol{A}(P) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} =$$

$$= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right)\boldsymbol{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right)\boldsymbol{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)\boldsymbol{k} = \operatorname{rot} \boldsymbol{A}(P).$$

Wektor funksiýa bolan bu funksiýa Gamilton operatory, kähalatlarda bolsa Gamiltonian diýilýär. Bu wektor funksiýa arkaly wektor algebranyň usullaryny ulanmak bilen gradiýentiň, diwergensiýanyň we rotoryň kesgitlemelerini alýarys. Bularda birinji tertipli önümleriň ulanylýandygyna görä, ol birinji tertipli wektor differensial operatorlardyr.

1-nji mysal. Eger $u = u(x, y, z)$ skalýar meýdan, $\boldsymbol{A}$=$\boldsymbol{A}(P)$ wektorly meýdan bolsa, $\operatorname{div}(u\boldsymbol{A})$ hasaplamaly.

Çözülişi. ∇ operatoryň ulanylýan köpeldijisini “$\downarrow$” belgi görkezýän bolsun. Onda $\operatorname{div}(u\boldsymbol{A})$-ny aşakdaky görnüşde ýazyp bileris:

$$(\nabla, u\boldsymbol{A}) = (\nabla, \overset{\downarrow}{u}\boldsymbol{A}) + (\nabla, u\,\overset{\downarrow}{\boldsymbol{A}}).$$

∇ operatory ulanylmaýan köpeldijini, skalýar köpeltmek hasylynyň daşyna çykaryp bolýandygynyň esasynda ýazyp bileris:

$$(\nabla, uA) = (\nabla, \overset{\downarrow}{u}A) + (\nabla, u\overset{\downarrow}{A}) = (\nabla u, A) + u(\nabla, A).$$

ýa-da

$$\operatorname{div}(u\boldsymbol{A}) = (\boldsymbol{A}, \operatorname{grad} u) + \boldsymbol{u}\operatorname{div}\boldsymbol{A}.$$

2-nji mysal. Eger u we v skalýar meýdanlar bolsa

$$\operatorname{grad}(\boldsymbol{uv}) = \boldsymbol{v}\operatorname{grad} u + \boldsymbol{u}\operatorname{grad} v$$

deňligi subut etmeli.

Çözülişi. $\operatorname{grad}(uv)$ aňlatmany başgaça ∇uv görnüşde ýazyp, alýarys:

$$\nabla \boldsymbol{uv} = \nabla \overset{\downarrow}{\boldsymbol{u}} v + \nabla \boldsymbol{u} \overset{\downarrow}{v} = \boldsymbol{v}\nabla \boldsymbol{u} + \boldsymbol{u}\nabla \boldsymbol{v}$$

ýa-da

$$\operatorname{grad}(\boldsymbol{uv}) = \boldsymbol{v}\operatorname{grad} u + \boldsymbol{u}\operatorname{grad} v.$$

135. Eger u skalýar meýdan, $\boldsymbol{A}(P)$ wektorly meýdan bolsa

$$\operatorname{rot}(u\boldsymbol{A}(P)) = \left[\operatorname{grad} u, \boldsymbol{A}(P)\right] + u\operatorname{rot}\boldsymbol{A}(P)$$

deňligi subut etmeli.

136. $\nabla(\frac{u}{v}) = \frac{v\nabla u - u\nabla v}{v^2}$ deňligi subut etmeli.

137. $\nabla^2(uv) = u\nabla^2 v + v\nabla^2 u + 2(\nabla u, \nabla v)$ deňligi subut etmeli. Bu ýerde $\nabla^2 = (\nabla, \nabla) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

§ 2. Ikinji tertipli differensial operatorlar

Skalýar ýa-da wektor meýdanlarda ∇ operatory iki gezek ulansak, biz ikinji tertipli differensial operatory alýarys.

Eger $\boldsymbol{u}(P)$ skalýar meýdan berlen bolsa, biz şu meýdanyň grad$\boldsymbol{u}$ tapyp bileris. Gradiýent meýdan bolsa wektor meýdandyr we biz şu meýdanyň diwergensiýasyny $(\operatorname{div}\operatorname{grad} u)$ we rotoryny $(\operatorname{rot}\operatorname{grad} u)$ tapyp bileris.

Eger $\boldsymbol{A}(P) = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}$ wektorly meýdan berlen bolsa, onda ondan iki meýdan emele gelýär: div $\boldsymbol{A}(P)$ skalýar meýdan we rot $\boldsymbol{A}(P)$

wektorly meýdan. Şunlukda, biz birinji meýdanyň $\operatorname{grad}\operatorname{div}\boldsymbol{A}(P)$ we ikinji meýdanyň $\operatorname{div}\operatorname{rot}\boldsymbol{A}(P)$ diwergensiýasyny we $\operatorname{rot}\operatorname{rot}\boldsymbol{A}(P)$ rotoryny tapyp bileris. Umuman biz bäş sany ikinji tertipli wektor differensial operatora seredip bileris. Esasan hem olaryň ilkinji üçüsi köp ulanylýar. Olara giňişleýin garap geçeliň.

1) $\operatorname{div}\operatorname{grad} u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$.

Hakykatdan-da, $\operatorname{grad} u = \dfrac{\partial u}{\partial x}\boldsymbol{i} + \dfrac{\partial u}{\partial y}\boldsymbol{j} + \dfrac{\partial u}{\partial z}\boldsymbol{k}$. Bu wektoryň diwergensiýasyny tapalyň:

$$\operatorname{div}\operatorname{grad} u = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

Deňligiň sag bölegindäki operatora Laplas operatory diýilýär we Δu bilen belgilenýär:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

Bu operatora indiki bölümçede giňişleýin serederis.

$$\operatorname{div}\operatorname{grad} u = \nabla(\nabla u) = \nabla^2 u.$$

2) $\operatorname{rot}\operatorname{grad} u = 0$.

Bu deňlik aňsat barlanylýar. Rotoryň aňlatmasyndaky her bir ýaý garyşyk birmeňzeş önümleriň tapawudyna deňdir. Mysal üçin,

$$\operatorname{rot}_x \operatorname{grad} u = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y}\right) = 0.$$

Eger bu deňligi ∇ wektorýň kömegi bilen ýazsak, onda ol aňsat ýadyňda galýar:

$$\operatorname{rot}\operatorname{grad} u = \nabla \times \nabla u = \left[\nabla \times \nabla\right] u = 0.$$

3) $\operatorname{rot}\boldsymbol{A}(P)$ wektordan diwergensiýany düzeliň:

$$\operatorname{div}\operatorname{rot}\boldsymbol{A}(P) = \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) =$$

$$=\frac{\partial^2 A_z}{\partial x\partial y}-\frac{\partial^2 A_y}{\partial x\partial z}+\frac{\partial^2 A_x}{\partial y\partial z}-\frac{\partial^2 A_z}{\partial y\partial x}+\frac{\partial^2 A_y}{\partial z\partial x}-\frac{\partial^2 A_x}{\partial z\partial y}=0.$$

Subut eden deňligimizi ∇ wektoryň kömegi bilen ýazalyň:

$$\text{div rot } \boldsymbol{A}(P)=\nabla\left[\nabla\times\boldsymbol{A}\right].$$

Biz üç wektoryň garyşyk köpeltmek hasylyny alýarys.

Beýleki iki birmeňzeş wektor operatorlar amaly meselelerde seýrek ulanylýar. Şeýle-de bolsa biz olaryň aralygyndaky baglanyşygy görkezeliň:

$$\text{rotrot}\boldsymbol{A}(P)=\text{graddiv}\boldsymbol{A}(P)=\text{graddiv}\boldsymbol{A}(P)-\Delta\boldsymbol{A}(P).$$

Bu ýerde Δ Laplas operatory, $\Delta\boldsymbol{A}(P)=\Delta A_x\boldsymbol{i}+\Delta A_y\boldsymbol{j}+\Delta A_z\boldsymbol{k}.$

1-nji mysal. Eger $r=\sqrt{x^2+y^2+z^2}$ bolsa, $\text{div}(\text{grad} f(r))$ tapmaly.

Çözülişi. Eger $f(r)$ differensirlenýan bolsa, onda $\text{grad} f(r)=\frac{f'(r)}{r}r$. Goý, indi $f(r)$ iki gezek differensirlenýän bolsun.

Amatlylyk üçin $\varphi(r)=\frac{f'(r)}{r}$ belgilemäni girizip

$\text{grad} f(r)=\frac{f'(r)}{r}r=\varphi(r)r$ görnüşde ýazalyň. Onda

$$\text{div}(\text{grad} f(r))=(\nabla,\varphi(r)r)=(r,\nabla\varphi(r))+\varphi(r)(\nabla,r)=$$
$$=(r,\text{grad}\varphi(r))+\varphi(r)\text{div}r=(r,\text{grad}\varphi(r))+3\varphi(r).$$

Bu ýerde $\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$ bolanlygy üçin $\text{div}\boldsymbol{r}=3$. Onda $\text{grad}\varphi(r)=\frac{\varphi'(r)}{r}r$ we $(r,\text{grad}\varphi(r))=r\varphi'(r)$ deňligiň esasynda alarys:

$$\text{div}(\text{grad} f(r))=r\varphi'(r)+3\varphi(r)=f'(r)-\frac{f'(r)}{r}+3\frac{f'(r)}{r}=f'(r)+$$
$$+\frac{2}{r}f(r).$$

138. Tapmaly

a) div($u$grad$u$);

b) div ($u$grad$v$).

139. Eger $r=\sqrt{x^2+y^2+z^2}$, $\boldsymbol{A}(P)$ we $\boldsymbol{B}(P)$ wektorly meýdanlar bolsa,

$\text{rot}\left[\boldsymbol{A}(P)\text{rot}\boldsymbol{B}(P)\right]$ hasaplamaly.

140. $\boldsymbol{A}(P)=x^2y^2\boldsymbol{i}+y^2z^2\boldsymbol{j}+z^2x^2\boldsymbol{k}$ wektorly meýdan bolsa, rot rot $\boldsymbol{A}(P)$ we grad div $\boldsymbol{A}(P)$ hasaplamaly.

141. Eger $u=u(x,y,z)$ funksiýanyň ikinji tertipli hususy önümleri bar bolsa rot grad u=O bolýandygyny subut etmeli.

§ 3. Laplas operatory

Goý, $u(x,y,z)$ funksiýanyň x, y, z ululyklara görä ikinji tertipli hususy önümleri bar bolsun. Onda

$$(\nabla,\nabla u)=(\frac{\partial}{\partial x}\boldsymbol{i}+\frac{\partial}{\partial y}\boldsymbol{j}+\frac{\partial}{\partial z}\boldsymbol{k},\frac{\partial u}{\partial x}\boldsymbol{i}+\frac{\partial u}{\partial y}\boldsymbol{j}+\frac{\partial u}{\partial z}\boldsymbol{k})=$$

$$\frac{\partial}{\partial x}(\frac{\partial u}{\partial x})+\frac{\partial}{\partial y}(\frac{\partial u}{\partial y})+\frac{\partial}{\partial z}(\frac{\partial u}{\partial z})=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\Delta u.$$

Şunlukda,

$$(\nabla,\nabla u)=\Delta u,\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\Delta u \tag{1}$$

aňlatma Gamilton operatory diýilýär. Oňa Gamilton operatoryny öz-özüne skalýar köpeltmek hasyly görnüşinde garamak bolar

$$\Delta=(\nabla,\nabla)=\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}.$$

Bu operator matematiki fizika dersinde örän wajyp orny tutýar. Bilşimiz ýaly, jisimde temperaturanyň ýaýramagynyň deňlemesi

$$\frac{\partial u}{\partial t}=a^2\Delta u,a^2=\frac{k}{c\rho}. \tag{2}$$

görnüşdedir. Eger-de jisim deňagramly ýylylyk ýagdaýynda, ýagny jisimde we onuň çäklerinde wagtyň geçmegi bilen temperatura üýtgemese $\frac{\partial u}{\partial t} = 0$ bolar we (2) deňleme $\Delta u = 0$ ýa-da

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{3}$$

görnüşi alar.

(3) deňlemä Laplas deňlemesi diýilýär. Bu deňleme bilen diňe temperaturanyň stasionar dargadylmagy ýazylman, Laplas deňlemesi bilen geçirijiniň üstünde zarýadlaryň deňölçegli dargadylmagy, ýapyk gapda gysylmaýan suwuklygyň durnukly hereketi we şuňa meňzeş hadysalar ýazylýar.

$\Delta u = 0$ deňlemäni kanagatlandyrýan $u(x, y, z)$ skalýar funksiýa garmonik funksiýa, degişli meýdana bolsa garmoniki meýdan diýilýär.

1-nji mysal. Eger $r = \sqrt{x^2 + y^2 + z^2}$ we Δ Laplas operatory bolsa, $\Delta(\frac{1}{r})$ hasaplamaly.

Çözülişi.

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{\partial r}{\partial x} = -\frac{1}{r^2}\frac{x}{\sqrt{x^2+y^2+z^2}} = -\frac{1}{r^2}\frac{x}{r},$$

$$\frac{\partial}{\partial y}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{y}{r}, \; \frac{\partial}{\partial z}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{z}{r}.$$

Onda

$$\Delta\left(\frac{1}{r}\right) = \frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right) = \frac{\partial}{\partial x}\left(-\frac{1}{r^2}\frac{x}{r}\right) +$$

$$+\frac{\partial}{\partial y}\left(-\frac{1}{r^2}\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(-\frac{1}{r^2}\frac{z}{r}\right) = -\frac{r^3 - 3r^2\frac{x}{r}x}{r^6} - \frac{r^3 - 3r^2\frac{y}{r}y}{r^6} -$$

$$-\frac{r^3-3r^2\frac{z}{r}z}{r^6}-\frac{3r^3-3r^3}{r^6}=0,\ r\neq 0.$$

Bu bolsa $u=\frac{1}{r},\ r\neq 0$ funksiýanyň garmonik funksiýadygyny görkezýär.

2-nji mysal. Eger

$$\nabla=\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k \text{ we } \Delta=\nabla^2=(\nabla,\nabla)=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

bolsa,

$$\nabla^2(u,v)=u\Delta v+v\Delta u+2\nabla u\nabla v$$

deňligi subut etmeli.

Çözülişi.

$$\nabla^2(u,v)=\nabla(\nabla,uv)=\nabla(\boldsymbol{i}\frac{\partial(uv)}{\partial x}+\boldsymbol{j}\frac{\partial(uv)}{\partial y}+\boldsymbol{k}\frac{\partial(uv)}{\partial z})=$$

$$=\nabla\left[v\left(\frac{\partial u}{\partial x}\boldsymbol{i}+\frac{\partial u}{\partial y}\boldsymbol{j}+\frac{\partial u}{\partial z}\boldsymbol{k}\right)+u\left(\frac{\partial v}{\partial x}\boldsymbol{i}+\frac{\partial v}{\partial y}\boldsymbol{j}+\frac{\partial v}{\partial z}\boldsymbol{k}\right)\right]=$$

$$=\left(\frac{\partial}{\partial x}\boldsymbol{i}+\frac{\partial}{\partial y}\boldsymbol{j}+\frac{\partial}{\partial z}\boldsymbol{k}\right)\left[v\left(\frac{\partial u}{\partial x}\boldsymbol{i}+\frac{\partial u}{\partial y}\boldsymbol{j}+\frac{\partial u}{\partial z}\boldsymbol{k}\right)+u\left(\frac{\partial v}{\partial x}\boldsymbol{i}+\frac{\partial v}{\partial y}\boldsymbol{j}+\frac{\partial v}{\partial z}\boldsymbol{k}\right)\right]=$$

$$=\frac{\partial v}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial z}\frac{\partial u}{\partial z}+\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}+$$

$$+v\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+u\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)=$$

$$=u\Delta v+v\Delta u+2\nabla u\nabla v.$$

142. Eger potensial meýdanyň $\boldsymbol{A}(P)$ wektorynyň diwergensiýasy $\rho(x,y,z)$ deň bolsa, bu meýdanyň $u(x,y,z)$ potensialynyň

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\Delta u=\rho(x,y,z)$$

Puasson deňlemesini kanagatlandyrýandygyny subut etmeli.

143. Aşakdaky skalýar meýdanlaryň garmonik meýdanlardygyny görkezmeli:

1) $u = x^2 + 2xy - y^2$;

2) $u = x^2 y + y^2 z + z^2 x$;

3) $u = x^2 - y^2$.

144. $u = \ln \frac{1}{r}, \quad r = \sqrt{x^2 + y^2}, r \neq 0$ skalýar meýdanyň garmonik meýdandygyny görkezmeli.

JOGAPLAR

1 bölüm

1. Merkezi $(x_0, y_0, z\ )$ nokatda bolan ellipsoidler maşgalasy. **2.** Merkezi koordinatalar başlangyjynda bolan sferalar maşgalasy. **3.** Bir parametrli parallel tekizlikler maşgalasy **4.** $x^2 + y^2 - z = C$ paraboloidler maşgalasy. **5.** $x^2 + y^2 = Cxz$ paraboloidler maşgalasy. **6.** Parallel tekizlikler maşgalasy. **7.** $2x - y = C$ parallel gönüler maşgalasy. **8.** $y = Cx, C > 0$ gönüler toplumy. **9.** $y^2 = Cx$ parabolalar maşgalasy. **10.** $x^2 - y^2 = C$ giperbolalar maşgalasy. **11.** –2. **12.** $1 - \sqrt{3}$. **13.** $\frac{\partial u}{\partial l} = \cos\alpha + \cos\beta + \cos\gamma$. **14.** $\frac{3}{2\sqrt{5}}$.

15. $\frac{8}{3}$. **16.** $-\frac{3\sqrt{2}}{4}$. **17.** $-\frac{\cos(l,r)}{r^2}$. **18.** $\frac{\sqrt{15}}{5}$. **19.** $\frac{3\sqrt{21}}{7}$. **20.** $-\frac{2}{5}$. **21.** –2.

22. $\frac{\pi a^2}{\sqrt{a^2 + R^2}}$. **23.** $\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$. **24.** $\frac{2}{3}(\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})$. **25.** $3\boldsymbol{i} - 2\boldsymbol{j} - 6\boldsymbol{k}$.

26. $\left|\operatorname{grad} u\right| = 4\sqrt{1 + (\ln 2)^2}$. **27.** $-3\boldsymbol{i}+3\boldsymbol{k}$, $3\boldsymbol{i}-3\boldsymbol{j}$, $3\boldsymbol{j}-3\boldsymbol{k}$. **28.** $\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$. **29.** $\boldsymbol{k}$. **30.** $4\boldsymbol{i}-4\boldsymbol{j}-2\boldsymbol{k}$. **31.** $\varphi=0$. **32.** $\varphi=\pi$. **33.** $\varphi=0$. **34.** $\varphi = 0$. **35.** $\varphi = \frac{\pi}{2}$. **36.** $a=-\boldsymbol{i}-$ $-2\boldsymbol{j}+2\boldsymbol{k}$ wektoryň ugry boýunça 3-e deň. **37.** $\sqrt{137}$. **38.** Merkezi $M(a,b,c)$ nokatda, radiusy 1-e deň bolan sferada.

II bölüm

38. a. $\frac{a}{\sqrt{x_1^2 + y_1^2 + z_1^2}} - \frac{a}{\sqrt{x_0^2 + y_0^2 + z_0^2}}$. **39.** $x = \cos t, y = \sin t, z = bt$ öwrülme çyzygy. **40.** $\frac{x}{a_1} = \frac{y - C_1}{a_2} = \frac{x - C_2}{a_3}$. **41.** $\begin{cases} x^2 + y^2 + z^2 = C_1^2 \\ x + y + z = C_2 \end{cases}$

42. $x^2=C_1y$, $z=C_2$. **43.** $x^2=C_1z$, $y=C_2$. **44.** $xy=C_1$, $z=C_2$. **45.** $x=C_1$, $2y^2-z^2=C_2$.

46. $\frac{1}{x}-\frac{1}{y}=C_1$, $z=C_2$. **47.** $y^2+z^2=C_1$, $x=C_2$. **48.** $4\pi ke$. **49.** –3.

50. $\frac{\pi}{5}$. **51.** $\frac{3\pi}{8}$. **52.** π. **53.** $\pi R^2\gamma$. **54.** $\pi R^2 h$. **55.** $\frac{5}{3}$. **56.** $\frac{\pi}{6}$. **57.** $-\pi$.

58. 0. **59.** $\frac{1}{10}\pi R^2 H(3R^2+2H^2)$. **60.** 0. **61.** $\frac{3\pi}{16}$. **62.** $-\frac{7}{6}$. **63.** $2\pi R^2 H$.

64. π. **65.** $6\pi R$. **66.** 0. **67.** $\frac{3}{4}\pi$. **68.** $\sqrt{2}\pi$. **69.** π. **70.** $\frac{\pi R^4}{2}$. **71.** $-\frac{256}{3}\pi$.

72. 8π. **73.** $\mathrm{div}(\frac{r}{|r|})=\frac{2}{r}$. **74.** 2(*x*+*y*+*z*). **75.** 3*u* 76. *x*+*y*+*z*. **77.** (grad*u,* grad*u*)+$u\,\Delta u$, $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ Laplas operatory. **78.** (grad*u,* grad*v*)+ +$u\Delta v$. **79.** $\frac{2}{r}f(r)+f'(r)$. **80.** 84. **83.** 0. **84.** Solenoidal meýdan **85.** Solenoidal meýdan däldir. **86.** Solenoidal meýdan. **87.** Solenoidal meýdan. **88.** Solenoidal meýdan. **89.** Solenoidal meýdan däldir. **90.** $|r_N|-|r_B|$. **91.** $Ln\frac{r_2}{r_1}$. **93.** πR^2+R. **94.** $-\frac{4}{3}R^2$. **95.** $-\pi R^2$. **96.** a) 1, b) 1, ç) 1. **97.** – 1.

98. $\frac{1}{35}$. **99.** $\pi\, a(2c-b)$. **100.** π. **101.** $\frac{188}{21}\ln 2$. **102.** 2π. **103.** $\frac{3}{4}\pi\, ab(a^2+b^2)$.

104. -2π. **105.** $-\frac{\pi R^3}{3}$. **106.** –*i*–(2*x*–1)*j*. **107.** $3(z^2-x^2)\boldsymbol{j}$. **108.** $(x+y)\boldsymbol{k}$.

109. *i*+*j*+*k*. **110.** $4\boldsymbol{i}+6\boldsymbol{j}+2\boldsymbol{k}$. **113.** $-\left[(y-z)\boldsymbol{i}+(z-x)\boldsymbol{j}+(x-y)\boldsymbol{k}\right]$.

114. 2(*j*–*k*). **117.** Potensial meýdan. **118.** Potensial meýdan däldir.

119. $\varphi = xy^2 + \frac{z^2}{2} + C$. **120.** $A_1(P)$ *we* $A_2(P)$ Potensial meýdan. **121.** Potensial meýdan däldir. **122.** Potensial meýdan. **123.** Potensial meýdan däldir. **124.** Potensial meýdan **125.** Potensial meýdan. **126.** $-\frac{5}{2}$. **127.** 9.

128. $\varphi = x + xyz$. **129.** $\varphi = x^2 y - y^2 + xz$. **130.** $\varphi = \ln|x+y+z| + C$.

131. $\varphi = \operatorname{arctg}(xyz) + C$. **132.** $\varphi = \alpha x + \beta y + \gamma z + C$. **133.** $\varphi = xy + e^z + C$.

134. $\varphi = e^x \sin y + z + C$. **138.** a) $\operatorname{div}(u \operatorname{grad} u) = |\operatorname{grad} u|^2 + u\Delta u$.

b) $\operatorname{div}(u \operatorname{grad} v) = (\operatorname{grad} u, \operatorname{grad} v) + u\Delta v$.
139. $(\operatorname{rot} \boldsymbol{B}(P), \nabla)\boldsymbol{A}(P) - (\boldsymbol{A}, \nabla)\operatorname{rot} \boldsymbol{B} - \operatorname{rot} \boldsymbol{B}(P) \operatorname{div} \boldsymbol{A}(P)$.

140. $\operatorname{rotrot} \boldsymbol{A}(P) = 2(2zx - x^2)\boldsymbol{i} + 2(2xy - y^2)\boldsymbol{j} + 2(2yz - z^2)\boldsymbol{k}$.

143. 1. Garmonik meýdan 2. Garmonik meýdan däldir 3. Garmonik meýdan.

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